

# THE PHANTOM MENACE IN REPRESENTATION THEORY

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## 1. INTRODUCTION AND PREREQUISITES

Our principal goal in this overview is to explain and motivate the concept of a *phantom* in the representation theory of finite dimensional algebras. In particular, we will exhibit the key role of phantoms towards understanding how a full subcategory  $\mathcal{A}$  of  $\Lambda\text{-mod}$  is embedded into  $\Lambda\text{-mod}$ , in terms of maps leaving or entering  $\mathcal{A}$ . (Here  $\Lambda$  is a finite dimensional algebra over a field  $K$ , and  $\Lambda\text{-mod}$  denotes the category of all finitely generated left  $\Lambda$ -modules.) Very roughly speaking, phantoms serve the following dual purpose in this connection: On one hand, they represent an effective tool for systematically tackling the question of whether or not  $\mathcal{A}$  is functorially finite in  $\Lambda\text{-mod}$ , or – in less technical terms – whether all objects in  $\Lambda\text{-mod}$  have best (right/left) approximations in  $\mathcal{A}$ , in a sense to be made precise momentarily. On the other hand, the *effective*  $\mathcal{A}$ -phantoms of a given object  $X \in \Lambda\text{-mod}$  capture – within a *minimal* frame – a condensed picture of the relations of those objects in  $\mathcal{A}$  which have nontrivial homomorphisms to  $X$ .

For a preview of the central definitions, let  $\mathcal{A}$  be closed under finite direct sums and  $\mathcal{C}$  a subcategory of  $\mathcal{A}$ . Recall that  $f \in \text{Hom}_\Lambda(A, X)$  is called a  *$\mathcal{C}$ -approximation of  $X$  inside  $\mathcal{A}$*  in case  $A$  belongs to  $\mathcal{A}$  and all maps in  $\text{Hom}_\Lambda(C, X)$  with  $C \in \mathcal{C}$  factor through  $f$  [29]. In case  $\mathcal{C} = \mathcal{A}$ , we re-encounter the classical right  $\mathcal{A}$ -approximations of  $X$  as introduced by Auslander-Smalø [4] and, independently, Enochs [19]. Whenever such right approximations exist, there is a unique candidate of minimal  $K$ -dimension, which thus serves as ‘best’ (right) approximation of  $X$  in  $\mathcal{A}$ . (The concept of left  $\mathcal{A}$ -approximation is dual; we will suppress the qualifier ‘right’ in the sequel, since we will leave dualization of our notions and results to the reader.) Note, however, that  $\mathcal{C}$ -approximations of  $X$  inside  $\mathcal{A}$  need not exist in general. One can, a priori, rely on their availability only when  $\mathcal{C}$  is finite. So if  $\mathcal{C}$  is countable, for instance, say  $\mathcal{C} = \{C_n \mid n \in \mathbb{N}\}$ , it is natural to consider  $\{C_1, \dots, C_n\}$ -approximations of  $X$  inside  $\mathcal{A}$ , and to explore whether they ‘converge’ to an object in  $\mathcal{A}$ . In general they will not, but rather keep growing in dimension. So the question arises whether one should drop the requirement that the expected limits be finitely generated. Of course, if we do not insist on *finitely generated* approximating objects, the factorization problem per se is trivialized; namely – to again refer to our countable test

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situation – forming the direct sum of a full collection of  $\{C_1, \dots, C_n\}$ -approximations of  $X$  inside  $\mathcal{A}$ , where  $n$  traces  $\mathbb{N}$ , will always yield the domain of a homomorphism through which all maps  $C_n \rightarrow X$  can be factored. Yet, such sums will be highly redundant with respect to this stipulation as a rule. The factorization problem thus re-gains interest if we require that the redundancy be trimmed off. The minimality condition which Happel and the author imposed [29] to produce condensed images of the ‘mapping behavior’ of  $\mathcal{A}$  relative to  $X$ , is as follows: An  $\mathcal{A}$ -*phantom* of  $X$  is a module  $H \in \Lambda\text{-Mod}$  such that each finitely generated submodule  $H'$  of  $H$  has the following property: there exists a finite subcategory  $\mathcal{A}' \subseteq \mathcal{A}$  – which will usually depend on  $H'$  – such that  $H'$  occurs as a subfactor of *every*  $\mathcal{A}'$ -approximation of  $X$  inside  $\mathcal{A}$ . On one hand, this definition is quite loose, in that the class of  $\mathcal{A}$ -phantoms of  $X$  is obviously closed under subfactors and direct limits (of directed systems). On the other hand, for *finite* classes  $\mathcal{A}' \subseteq \mathcal{A}$ , one expects a large number of  $\mathcal{A}'$ -approximations of  $X$  inside  $\mathcal{A}$ , whence the requirement that  $H'$  should make an appearance in all of them places major pressure on  $H'$ . The slack in the definition is useful towards testing whether  $X$  has a classical  $\mathcal{A}$ -approximation, for the following reason: The existence of classical approximations obviously forces all  $\mathcal{A}$ -phantoms of  $X$  to be subfactors of the minimal  $\mathcal{A}$ -approximation of  $X$ . In fact, the class of phantoms contains infinite dimensional objects – or, equivalently, objects of unbounded finite dimensions – if and only if  $X$  fails to have a classical  $\mathcal{A}$ -approximation. So the easier it is to construct (infinite dimensional) phantoms, the better, as long as they are only to serve as indicators for the classical existence problem. On the other hand, in case this problem has already been resolved in the negative, small phantoms are no longer of interest. One is led to investigate a class of upgraded phantoms that are fairly close in spirit to the classical approximations, except for usually being infinite dimensional: namely, the ‘effective’ phantoms. Given a subcategory  $\mathcal{C} \subseteq \mathcal{A}$ , a  $\mathcal{C}$ -*effective phantom*  $H$  of  $X$  is a direct limit of a directed system of objects in  $\mathcal{A}$  which is a phantom and comes equipped with a homomorphism  $H \rightarrow X$ , through which all homomorphisms  $C \rightarrow X$  for  $C \in \mathcal{C}$  can be factored. For existence of effective phantoms, see Section 4.

On the side, we mention that, in algebraic topology, there has long been interest in *phantom maps*, namely in maps  $f : X \rightarrow Y$  of CW-complexes with the property that, for each  $n \in \mathbb{N}$ , the restriction of  $f$  to the  $n$ -skeleton  $X^n$  of  $X$  is null-homotopic (see [40] for a survey and [14] for a modified notion of a phantom map). Around the same time as Happel and the author introduced the concepts outlined above, Benson and Gnacadja [8] defined phantom maps in the context of group representations as follows: Given a finite group  $G$  and a field  $K$  of positive characteristic, a homomorphism  $X \rightarrow Y$  of  $KG$ -modules is said to be a phantom map if its restriction to every finitely generated submodule of  $M$  factors through a projective module. These phantom maps share a decisive feature with the universal maps accompanying the effective phantoms we mentioned in the preceding paragraph: Namely, they are comparatively well understood on restrictions to finitely generated subobjects of the domain, while globally, they often display qualitatively new properties.

Let us add a bit more history. The approximation theory that gave rise to the work recorded in this overview has its roots in the discovery that injective envelopes and pro-

jective covers are highly useful auxiliary objects in the structural analysis of more general modules. In their seminal paper of 1953, Eckmann and Schopf [18] showed that every module can be embedded as an essential submodule into an injective module, and in 1962, Gabriel [23] extended their findings to the context of Grothendieck categories. The more delicate existence question for projective covers, on the other hand, was resolved by Bass [6] in 1960, in his famous Theorem P. Put in slightly different terms, these results address best approximations of a module  $X$  by an object from the category of all injective, resp. projective, modules. The passage to more general subcategories of modules was performed by Auslander-Smalø [4] and Enochs [19], almost simultaneously, in the early eighties. The former team was primarily interested in categories of finitely generated modules over Artin algebras and spoke about right and left approximations, while Enochs focused on more general settings, such as coherent or noetherian rings, calling the decisive maps covers and envelopes, depending on whether they leave or enter the considered module category. The artinian line was first used towards sufficient conditions for the existence of almost split sequences in subcategories by Auslander and Smalø [5] in 1981; a decade later, this line obtained an important boost through its applications to homology and tilting theory, first observed by Auslander-Reiten [3]. Meanwhile, the latter approach was pursued separately by Asensio Mayor, Belshoff, Enochs, Guil-Asensio, Martínez Hernández, Rada, Saorín, del Valle, Xu, and others (see [2, 7, 20, 39, 41, 42, 52, 53]). Finally, we mention that in [38], Levy independently used minimal approximations from a category of particularly accessible objects, in order to study modules over pullback rings.

In the sequel, our favorite category will be  $\mathcal{P}^\infty(\Lambda\text{-mod})$ , the category of all finitely generated left  $\Lambda$ -modules of finite projective dimension; it is contained in the category  $\mathcal{P}^\infty(\Lambda\text{-Mod})$  having as objects *all*  $\Lambda$ -modules of finite projective dimension. To place the new concepts (of Section 4) into context, we will discuss and exemplify (in Sections 2,3) the massive impact of contravariant finiteness of  $\mathcal{P}^\infty(\Lambda\text{-mod})$  on the representation theory of  $\Lambda$ . The most striking results in this direction are Theorems 2 and 3 of Section 3 which, in case of existence, identify the minimal  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximations of the simple modules as the basic building blocks of arbitrary objects in  $\mathcal{P}^\infty(\Lambda\text{-Mod})$  – and as the indicators of little and big finitistic dimensions; these theorems combine results of Auslander and Reiten [3] with more recent work of Smalø and the author [35]. The resulting program of accessing the structure of objects in  $\mathcal{P}^\infty(\Lambda\text{-Mod})$  can be carried out in an ideally explicit format when  $\Lambda$  is left serial [12]. This first model situation will be presented at the end of Section 3. The second class of algebras for which the theory developed yields particularly complete and smooth results is the class of string algebras. In that case, answers to essentially all homological questions one might pose can be provided in the form of a finite number of ‘characteristic’  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -phantoms, one for each of the simple left modules. This is the content of ongoing joint work of the author with Smalø [36]; it is sketched in Section 5. We include a brief history of the theory of string algebras, since these algebras have established their role as excellent display cases of the representation-theoretic methods developed during the past decades.

Since our primary goal is to familiarize the reader with the type of information stored in phantoms, and since we consider examples as one of the keys to an intuitive grasp, there

will be major emphasis on explicit illustrations. In fact, we start by setting up a sequence of ‘test examples’ to which we keep returning as the discussion proceeds.

This is a revised and updated version of a survey that appeared in the Proceedings of the 30th Symposium on Ring Theory and Representation Theory, which took place in Nagano, Japan, in 1997. The author would like to thank the organizer/editor, Y. Iwanaga, for his permission to incorporate portions of that earlier publication here. Moreover, the author is indebted to M. Saorín for references to the history of the non-artinian thread of the subject.

*There are people indeed [ . . . ] for whom all the things that have a fixed value, assessable by others, fortune, success, high positions, do not count; what they must have is phantoms.*

Marcel Proust, *Remembrance of Things Past*

### Content overview.

2. Contravariant finiteness and first examples
3. Homological importance of contravariant finiteness and a model application of the theory
4. Phantoms. Definitions, existence, and basic properties
5. Phantoms over string algebras – another model setting

### Prerequisites.

Throughout,  $\Lambda = K\Gamma/I$  will be a path algebra modulo relations with Jacobson radical  $J$ , and the vertex set of  $\Gamma$  will be identified with a full set of primitive idempotents of  $\Lambda$ . By  $\Lambda\text{-Mod}$  we will denote the category of all left  $\Lambda$ -modules and by  $\Lambda\text{-mod}$  the full subcategory of finitely generated modules. Moreover,  $\mathcal{P}^\infty(\Lambda\text{-Mod})$  and  $\mathcal{P}^\infty(\Lambda\text{-mod})$  will be the subcategories of  $\Lambda\text{-Mod}$  and  $\Lambda\text{-mod}$ , respectively, having as objects the modules of finite projective dimension. The suprema of the projective dimensions attained on  $\mathcal{P}^\infty(\Lambda\text{-Mod})$  and  $\mathcal{P}^\infty(\Lambda\text{-mod})$  will be labeled  $\text{l Fin dim } \Lambda$  and  $\text{l fin dim } \Lambda$ , respectively.

Given a path  $p$  in  $K\Gamma$ , we will denote by  $\text{start}(p)$  and  $\text{end}(p)$  the starting and end points of  $p$ , respectively, and our convention for multiplying paths  $p, q \in K\Gamma$  is as follows: “ $qp$ ” stands for “ $q$  after  $p$ ”.

Our most important auxiliary devices will be labeled and layered graphs of finite dimensional modules. Since our graphing conventions differ to some extent from those of other authors (in particular, they are akin to, but not the same as, the module diagrams studied by Alperin [1] and Fuller [22]), we include an informal description of our graphs for the convenience of the reader.

The graphs we use are based on sequences of ‘top elements’ of a module  $M$  which are  $K$ -linearly independent modulo  $JM$ . Here  $x \in M$  is a *top element* of  $M$  if  $x \notin JM$  and  $x = ex$  for one of the primitive idempotents  $e$  corresponding to the vertices of  $\Gamma$ ; in this situation we will also say that  $x$  is a top element of *type*  $e$ .

Let  $\Lambda = K\Gamma/I$  be the algebra presented in Example C.1 of Section 2 below. That the indecomposable projective module  $\Lambda e_6$  have graph

$$\begin{array}{c}
6 \\
\rho \downarrow \quad \searrow \sigma \\
5 \qquad 5 \\
\delta \downarrow \quad \searrow \epsilon \\
5 \qquad 5
\end{array}$$

with respect to the top element  $e_6$  means that  $J^3 e_6 = 0$  and  $Je_6/J^2 e_6 \cong J^2 e_6 \cong S_5 \oplus S_5$ , that  $\rho e_6$  and  $\sigma e_6$  are  $K$ -linearly independent modulo  $J^2 e_6$ , and that  $\delta \rho e_6$  and  $\epsilon \rho e_6$  are  $K$ -linearly independent (modulo  $J^3 e_6$ ), while  $\delta \sigma e_6 = \epsilon \delta e_6 = 0$  in  $\Lambda e_6$ .

Generally, the entries in each row of a graph of a module  $M$  record the composition factors in the radical layers  $J^r M/J^{r+1} M$  in the correct multiplicities. Whenever we present the graph of an indecomposable projective module  $\Lambda e$ , we tacitly assume that the corresponding top element of  $\Lambda e$  is chosen to be  $e$ . In our first example the choice of top element of  $\Lambda e_6$  does not influence the graph, but it will in other situations. For instance, the module  $M = (\Lambda e_6 \oplus \Lambda e_6)/U$ , where  $U = \Lambda(\rho e_6, \rho e_6)$ , has graph

$$\begin{array}{ccccc}
& x_1 & & x_2 & \\
& 6 & & 6 & \\
\sigma \swarrow & & \rho & \rho & \searrow \sigma \\
5 & & 5 & & 5 \\
& \delta \swarrow & & \searrow \epsilon & \\
& 5 & & 5 &
\end{array}$$

relative to the top elements  $x_1 = (e_6, 0)$  and  $x_2 = (0, e_6)$ , while its graph relative to the top elements  $y_1 = x_1 + x_2$  and  $y_2 = x_2$  is

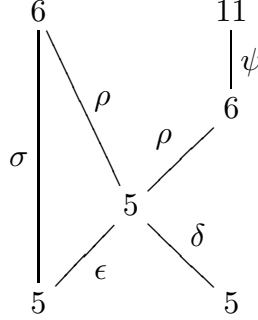
$$\begin{array}{ccc}
y_1 & & y_2 \\
6 & & 6 \\
\sigma \downarrow & \oplus & \rho \swarrow \quad \searrow \sigma \\
5 & & 5 \qquad 5 \\
& \delta \swarrow \quad \searrow \epsilon & \\
& 5 \qquad 5 &
\end{array}$$

Note that the graph of a module need not determine the module in question, up to isomorphism. For example, each of the modules  $M_k = \Lambda e_6/U_k$ , where  $U_k = \Lambda(\sigma - k\rho)e_6$  with  $k \in K^*$ , has graph

$$\begin{array}{c}
6 \\
\sigma \left( \begin{array}{c} \phantom{0} \\ \phantom{0} \\ \phantom{0} \end{array} \right) \rho \\
5
\end{array}$$

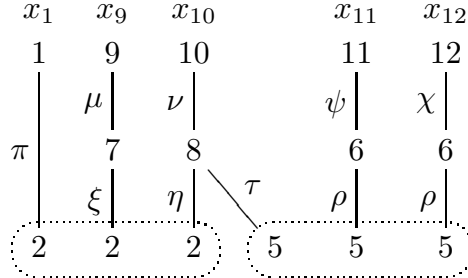
with respect to the top element  $e_6 + U_k$  of  $M_k$ , while  $M_{k_1} \not\cong M_{k_2}$  if  $k_1 \neq k_2$ .

For a slightly more involved example, consider the factor module  $N = (\Lambda e_6 \oplus \Lambda e_{11})/V$ , where  $V$  is the submodule generated by  $(\rho, \rho\psi + \delta\rho\psi)$  and  $(\epsilon\rho - \sigma, 0)$ . Relative to the top elements  $x_1 = (e_6, 0)$  and  $x_2 = (0, e_{11})$ , the module  $N$  has graph



indicating – next to the previously discussed information – that  $\sigma x_1$  is equal to a nonzero scalar multiple of  $\epsilon\rho x_1$ , modulo  $J^4M = 0$ , and  $\rho x_1$  equals a nonzero scalar multiple of  $\rho\psi x_2$ , modulo  $J^3M$ .

To enlarge the family of objects in  $\Lambda$ -mod which possess labeled and layered graphs relative to suitable sequences of top elements, we also allow for graphs with ‘pools’ along the following model. That a module  $A \in \Lambda$ -mod has graph



is to encode the following information: The module  $A$  is generated by top elements  $x_i$  of type  $e_i$  ( $i = 1, 9, 10, 11, 12$ ) which are  $K$ -linearly independent modulo  $JA$  (here automatically satisfied, since  $e_i \neq e_j$  for  $i \neq j$ ) such that  $J^3A = 0$ , and

(a)  $JA/J^2A \cong S_7 \oplus S_8 \oplus S_6^2$ , with  $\mu x_9, \nu x_{10}, \psi x_{11}, \chi x_{12}$  being  $K$ -linearly independent modulo  $J^2A$ ;

(b)  $J^2A \cong S_2^2 \oplus S_5^2$ , and the “pooled elements”  $\pi x_1, \xi\mu x_9, \eta\nu x_{10}$  are  $K$ -linearly dependent, while any two of these elements are  $K$ -linearly independent; analogously,  $\tau\nu x_{10}, \rho\psi x_{11}, \rho\chi x_{12}$  are  $K$ -linearly dependent, with any subset of two  $K$ -linearly independent.

Note that in this particular example, the module  $A$  is determined up to isomorphism by its graph, since the coefficients arising in the mentioned linear dependence relations can be adjusted by suitably modifying the top elements by scalar factors.

It is clear that we will not lose information if we omit the labels on edges

$$\begin{array}{c} i \\ | \\ j \end{array}$$

with the property that there is a unique arrow  $i \rightarrow j$  in  $\Gamma$ . Moreover, it should be self-explanatory that certain countably generated left  $\Lambda$ -modules can be communicated by graphs as well. The *infinite* graph

$$\begin{array}{ccccccc} 6 & & 6 & & 6 & & \dots\dots \\ & \searrow & \nearrow & \searrow & \nearrow & \searrow & \\ \sigma & & \rho & & \sigma & & \\ & \nearrow & \searrow & \nearrow & \searrow & \nearrow & \\ 5 & & 5 & & 5 & & \dots\dots \end{array}$$

for instance, goes with a module  $B$ , uniquely determined up to isomorphism, which is generated by top elements  $x_1, x_2, x_3, \dots$  of type  $e_6$  which are  $K$ -linearly independent modulo  $JB$  such that  $\rho x_1 = 0$  and  $\sigma x_i = \rho x_{i+1}$  for all  $i \geq 1$ , and such that the elements  $\sigma x_i$ ,  $i \in \mathbb{N}$ , are  $K$ -linearly independent (modulo  $J^2B = 0$ ).

## 2. CONTRAVARIANT FINITENESS AND FIRST EXAMPLES

In 1980, Auslander and Smalø [4] introduced the following definitions – along with the duals – in connection with their search for conditions ensuring existence of almost split sequences. It turned out that existence is guaranteed in full, extension-closed subcategories  $\mathcal{A}$  of  $\Lambda\text{-mod}$  which are both co- and contravariantly finite in the sense recalled below. Even though their results have been applied on numerous occasions during the 1980's, it was not until the 1990's that the concept of contravariant finiteness reached a high level of popularity, due to its links with homology and tilting. The spark in the tinder barrel was a paper by Auslander and Reiten [3]. We will describe their homological results in Section 3.

**Definitions.** Let  $\mathcal{A} \subseteq \Lambda\text{-mod}$  be a full subcategory and  $X \in \Lambda\text{-mod}$ .

(1) A *right  $\mathcal{A}$ -approximation* of  $X$  is a homomorphism  $f : A \rightarrow X$  with  $A \in \mathcal{A}$  such that each  $g \in \text{Hom}(B, X)$  with  $B \in \mathcal{A}$  factors through  $f$ , or equivalently, such that the following sequence of functors induced by  $f$  is exact:

$$\text{Hom}_\Lambda(-, A)|_{\mathcal{A}} \longrightarrow \text{Hom}_\Lambda(-, X)|_{\mathcal{A}} \longrightarrow 0.$$

(Since we will hardly mention the dual concept of ‘left approximation’, we will systematically suppress the qualifier ‘right’ when discussing approximations.)

(2) The subcategory  $\mathcal{A}$  is said to be *contravariantly finite* (in  $\Lambda\text{-mod}$ ) if each  $X \in \Lambda\text{-mod}$  has an  $\mathcal{A}$ -approximation, i.e., if each of the functors  $\text{Hom}_\Lambda(-, X)|_{\mathcal{A}}$  is finitely generated in the category of all contravariant functors from  $\mathcal{A}$  to  $\mathbf{Ab}$ .

Suppose that  $X$  has an  $\mathcal{A}$ -approximation. By a slight abuse of language, we will then also refer to the source of this map as an approximation. Not surprisingly, the  $\mathcal{A}$ -approximations of  $X$  of minimal length are all isomorphic. Indeed, as was shown by Auslander and Smalø [4], given a minimal  $\mathcal{A}$ -approximation  $f : A \rightarrow X$  and any  $\mathcal{A}$ -approximation  $f' : A' \rightarrow X$ , there exists a split embedding  $g : A \rightarrow A'$  which makes the following diagram commute:

$$\begin{array}{ccc} A & \xrightarrow{g} & A' \\ & \searrow f & \swarrow f' \\ & X & \end{array}$$

It is thus justified to refer to the minimal  $\mathcal{A}$ -approximation of  $X$  in case of existence.

If  $\mathcal{A}$  is a resolving subcategory of  $\Lambda\text{-mod}$ , i.e., if  $\mathcal{A}$  contains all projectives in  $\Lambda\text{-mod}$  and is closed under extensions and kernels of epimorphisms, then the simples play a prominent role in checking contravariant finiteness. Indeed:

**Theorem 1.** [3] *Suppose  $\mathcal{A}$  is a resolving subcategory of  $\Lambda\text{-mod}$ . Then  $\mathcal{A}$  is contravariantly finite in  $\Lambda\text{-mod}$  if and only if each of the simple left  $\Lambda$ -modules has an  $\mathcal{A}$ -approximation.  $\square$*

Since, clearly, our favorite category  $\mathcal{P}^\infty(\Lambda\text{-mod})$  is resolving, this will provide a convenient test for contravariant finiteness. As we will see in the next section, the minimal  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximations of the simples are the basic structural building blocks for the objects in  $\mathcal{P}^\infty(\Lambda\text{-Mod})$  in case of existence, which makes it a priority to understand the structure of these particular approximations.

To provide examples, we begin with some

### Well Known Facts.

- A subcategory  $\mathcal{A} \subseteq \Lambda\text{-mod}$  is contravariantly finite in case it is ‘very big’ or ‘very small’. Indeed, if  $\mathcal{A} = \Lambda\text{-mod}$ , then clearly contravariant finiteness is guaranteed, the minimal approximations being the identity maps. So, in particular,  $\mathcal{P}^\infty(\Lambda\text{-mod})$  is contravariantly finite provided that  $\Lambda$  has finite global dimension. On the other hand, if  $\mathcal{A}$  has finite representation type, i.e., if there exist objects  $A_1, \dots, A_n \in \mathcal{A}$  such that each object in  $\mathcal{A}$  is a direct sum of copies of the  $A_i$ , we have contravariant finiteness as well [3]. To construct an  $\mathcal{A}$ -approximation of a module  $X$ , simply add up as many copies of each  $A_i$  as the  $K$ -dimension of  $\text{Hom}_\Lambda(A_i, X)$  indicates. In particular, if  $\mathcal{A}$  is the category of all finitely generated projectives in  $\Lambda\text{-mod}$ , the minimal  $\mathcal{A}$ -approximations are precisely the projective covers. This latter category coincides with  $\mathcal{P}^\infty(\Lambda\text{-mod})$  precisely when  $\text{l fin dim } \Lambda = 0$ .

- [3] If  $\Lambda$  is stably equivalent to a hereditary algebra  $\Lambda'$  (meaning that the stable category  $\underline{\Lambda\text{-mod}}$ , obtained as a factor category from  $\Lambda\text{-mod}$  by killing the maps that factor through projectives, is equivalent to the corresponding stable category  $\underline{\Lambda'\text{-mod}}$ ), then  $\mathcal{P}^\infty(\Lambda\text{-mod})$  is contravariantly finite. This hypothesis is, in particular, satisfied if  $J^2 = 0$ , and in that case, the minimal approximation  $A(X)$  of any  $\Lambda$ -module  $X$  can be readily pinned down



as follows [32]:  $A(X) = P/(JP)_{\text{fin}}$ , where  $P$  is the projective cover of  $X$  and  $(JP)_{\text{fin}}$  is the direct sum of those homogeneous components of  $JP$  which have finite projective dimension.

- [12] If  $\Lambda$  is left serial, then  $\mathcal{P}^\infty(\Lambda\text{-mod})$  is always contravariantly finite. The minimal approximations of the simple left modules arising in this situation will be described in the next section.

- The first example for which  $\mathcal{P}^\infty(\Lambda\text{-mod})$  was shown not to be contravariantly finite is due to Igusa-Smalø-Todorov [37]. It is a monomial relation algebra with  $J^3 = 0$  and  $\text{l fin dim } \Lambda = 1$  which is closely related to the Kronecker algebra, its  $K$ -dimension exceeding that of the latter algebra only by 2. In this example, the right finitistic dimension is 0, which, in view of our first remark, demonstrates that the right-hand category  $\mathcal{P}^\infty(\text{mod-}\Lambda)$  is contravariantly finite in  $\text{mod-}\Lambda$ . Thus contravariant finiteness of  $\mathcal{P}^\infty(-)$  is not left-right symmetric.

- In [29], very general criteria for the failure of contravariant finiteness of  $\mathcal{P}^\infty(\Lambda\text{-mod})$  were developed. We refer the reader to the examples given there and in [34].

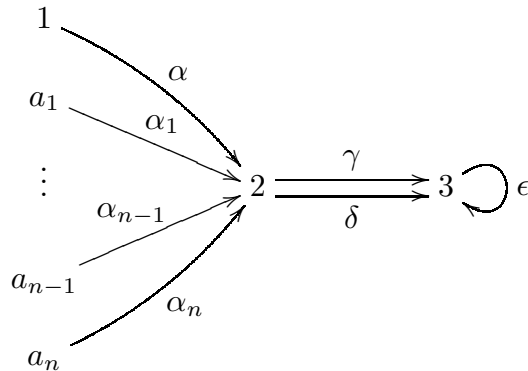
The emerging picture indicates that, while hardly any of the traditionally considered classes of finite dimensional algebras enjoy – en bloc – the property that  $\mathcal{P}^\infty(-)$  is contravariantly finite, the positive case is ubiquitous. In fact, the condition of having contravariantly finite  $\mathcal{P}^\infty(-)$  appears to slice diagonally through the prominent classes of finite dimensional algebras.

### First Installment of Nonstandard Examples.

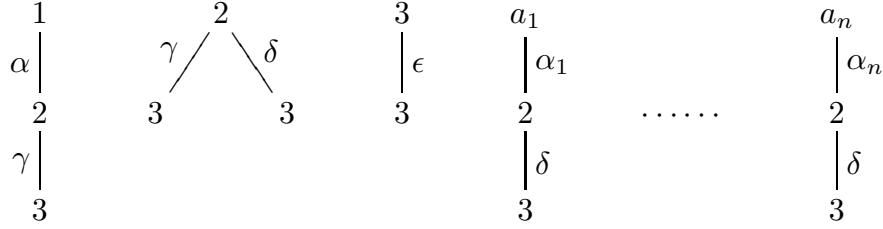
We will next present a first set of examples which are to informally communicate prototypical phenomena ensuring that a given simple module has a  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximation or fails to have such an approximation. These specific algebras will then continue to serve us as illustrations along the way. We will follow with proofs of some of the positive instances, but defer the discussion of the negative instances to Section 4.

**Examples A.** Our first example shows that, for each natural number  $n$ , there exists a finite dimensional monomial relation algebra  $\Lambda$  and a simple  $S \in \Lambda\text{-mod}$  such that  $\mathcal{P}^\infty(\Lambda\text{-mod})$  is contravariantly finite and the minimal  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximation of  $S$  is a direct sum of  $n$  distinct nonzero indecomposable components.

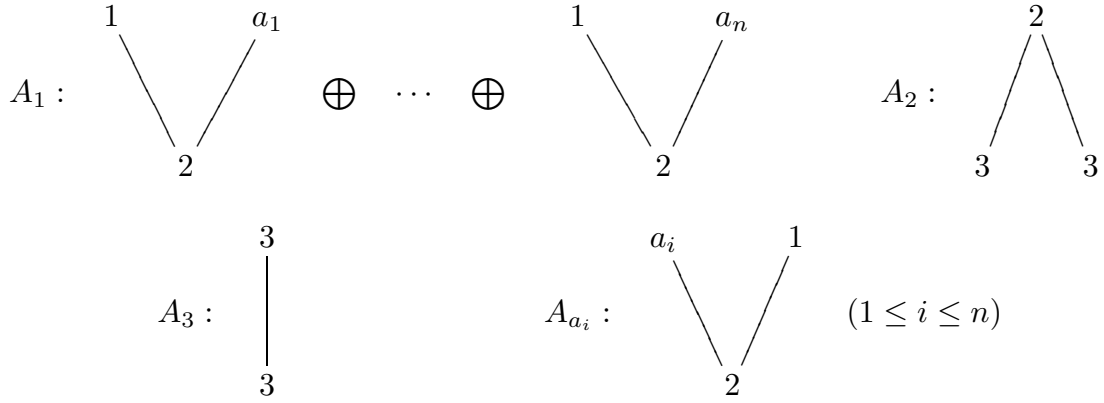
**A.1.** Fix  $n \in \mathbb{N}$ , and let  $\Lambda = K\Gamma/I$  be the monomial relation algebra with quiver  $\Gamma$



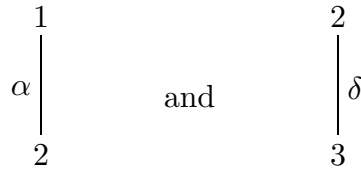
having indecomposable projective left modules with graphs



Then  $\mathcal{P}^\infty(\Lambda\text{-mod})$  is contravariantly finite, and the minimal  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximations  $A_1, A_2, A_3, A_{a_1}, \dots, A_{a_n}$  of the simple modules  $S_1, S_2, S_3, S_{a_1}, \dots, S_{a_n}$  have the following graphs (which determine the corresponding modules up to isomorphism).

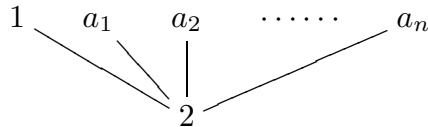


**A.2.** Now let  $\Lambda = K\Gamma'/I'$ , where  $\Gamma'$  is obtained from the quiver  $\Gamma$  of A.1 by removing the arrow  $\gamma$ , and the ideal  $I' \subseteq K\Gamma'$  is such that the graphs of  $\Lambda e_3, \Lambda e_{a_1}, \dots, \Lambda e_{a_n}$  are as under A.1, whereas  $\Lambda e_1$  and  $\Lambda e_2$  have graphs



respectively.

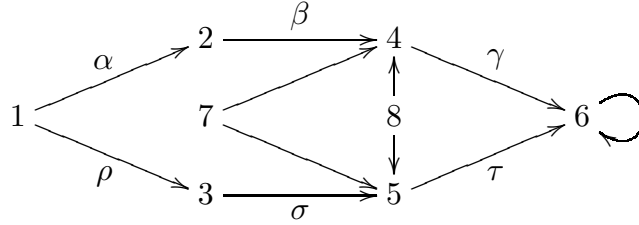
Then  $S_1 = \Lambda e_1/Je_1$  has minimal  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximation



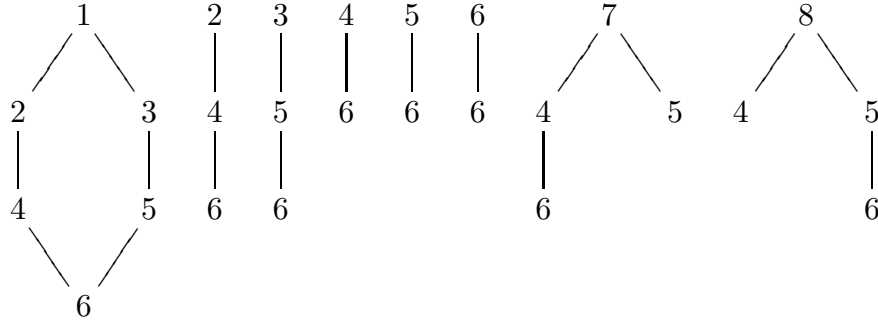
$S_2$  has minimal  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximation  $\Lambda e_2$ , and  $S_{a_1}, \dots, S_{a_n}$  belong to  $\mathcal{P}^\infty(\Lambda\text{-mod})$ . In particular,  $\mathcal{P}^\infty(\Lambda\text{-mod})$  is again contravariantly finite.

### Examples B.

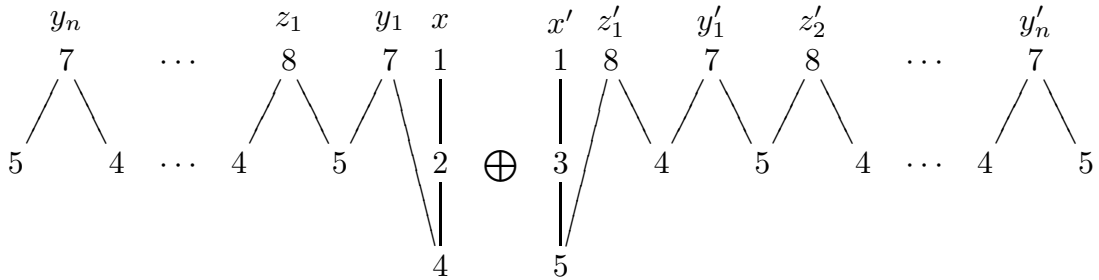
**B.1.** Let  $\Lambda = K\Gamma/I$ , where  $\Gamma$  is the quiver



and the ideal  $I \subseteq K\Gamma$  contains the relation  $\gamma\beta\alpha - \tau\sigma\rho$ , together with suitable monomial relations, such that the indecomposable projective left  $\Lambda$ -modules have graphs



Then  $S_i \in \mathcal{P}^\infty(\Lambda\text{-mod})$  for  $i = 2, 3$ , the minimal  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximations of  $S_4, S_5, S_6$  are  $\Lambda e_4, \Lambda e_5$ , and  $\Lambda e_6$ , respectively, while none of  $S_1, S_7, S_8$  has a  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximation. In fact, in Section 4, we will see that there is no object  $A \in \mathcal{P}^\infty(\Lambda\text{-mod})$  such that all of the homomorphisms from the modules in the following subclass of  $\mathcal{P}^\infty(\Lambda\text{-mod})$  – call it  $\mathcal{C}$  – to  $S_1$  factor through  $A$ .



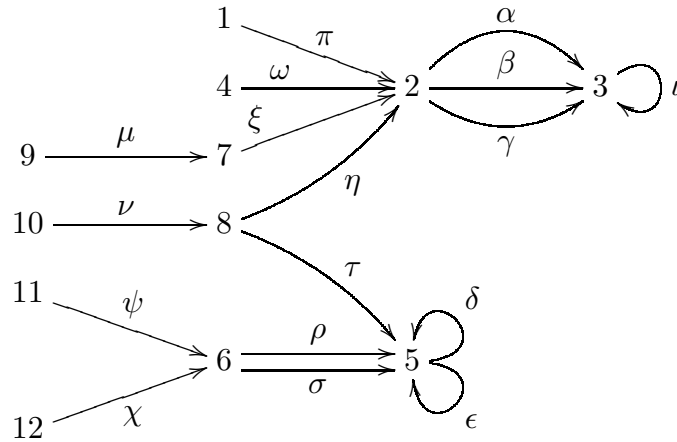
**B.2.** Now let  $\Lambda$  be the factor algebra of the algebra described in B.1, modulo the ideal generated by  $e_7$  and  $e_8$ . Then the graphs of the indecomposable projective modules  $\Lambda e_1, \dots, \Lambda e_6$  remain unchanged, but this time  $\mathcal{P}^\infty(\Lambda\text{-mod})$  is contravariantly finite. Indeed, the minimal  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximations of  $S_2, \dots, S_6$  are as above, while  $S_1$  has the following minimal  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximation:

$$\begin{array}{ccc} 1 & & 1 \\ | & & | \\ 2 & \oplus & 3 \\ | & & | \\ 4 & & 5 \end{array}$$

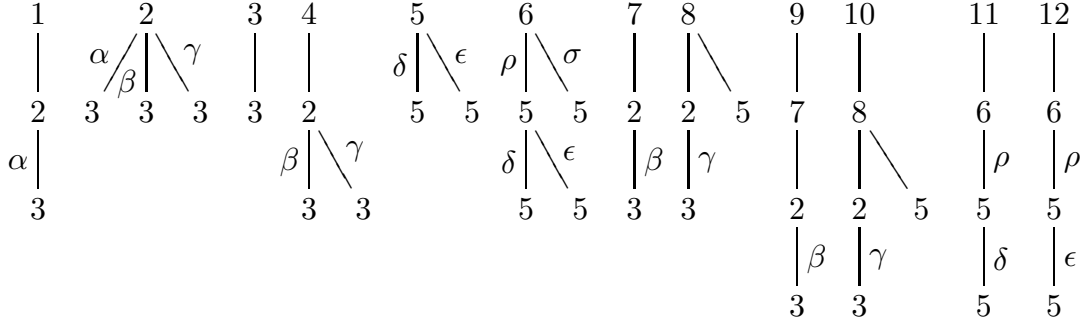
The next example shows that the structure of the minimal  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximations of the simple modules need by no means be as simplistic as in the previous instances, not even in situations where the indecomposable projective modules are of a simplistic makeup.

### Examples C.

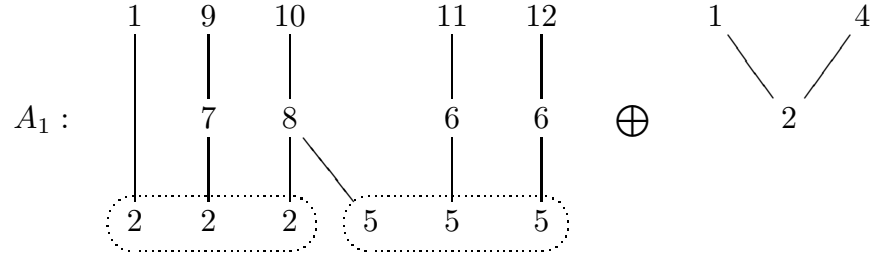
**C.1.** This time, let  $\Lambda = K\Gamma/I$  be the monomial relation algebra with quiver  $\Gamma$



and choose the ideal  $I \subseteq K\Gamma$  of relations so that the indecomposable projective left  $\Lambda$ -modules have the following graphs:

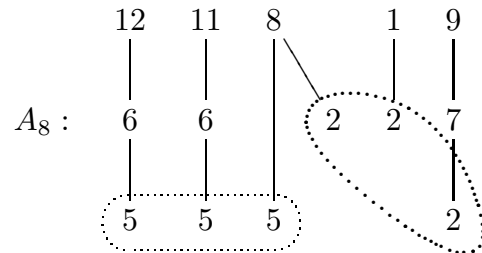
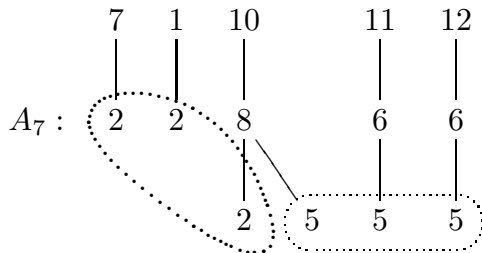
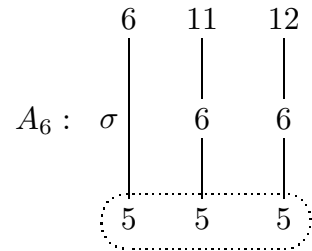
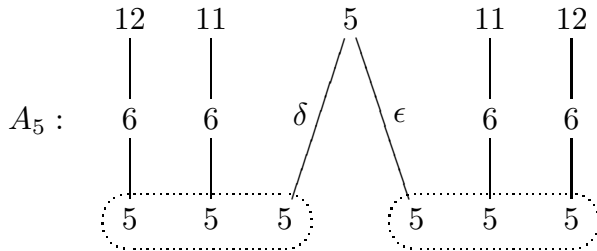
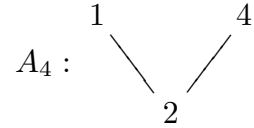


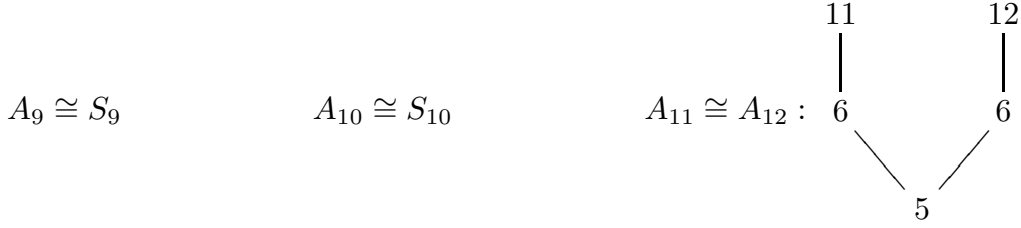
In this example,  $\mathcal{P}^\infty(\Lambda\text{-mod})$  is again contravariantly finite, and the minimal approximations  $A_1, \dots, A_{12}$  of the simples  $S_1, \dots, S_{12}$  are determined by their graphs as follows:



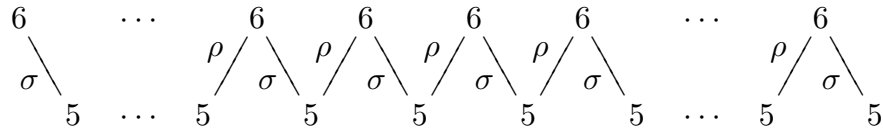
$$A_2 \cong \Lambda e_2$$

$$A_3 \cong \Lambda e_3$$





**C.2.** Finally, let  $\Lambda$  be the factor algebra of the algebra under C.1, modulo the ideal generated by  $e_{11}$  and  $e_{12}$ . Note that the graphs of the indecomposable projectives  $\Lambda e_1, \dots, \Lambda e_{10}$  remain the same as in C.1, since  $e_{11}$  and  $e_{12}$  are sources of  $\Gamma$ . This time,  $\mathcal{P}^\infty(\Lambda\text{-mod})$  fails to be contravariantly finite,  $S_1, S_5, S_6, S_8$  being precisely those simples which have lost their  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximations in the passage to the smaller algebra. As we will see in Section 4, the homomorphisms onto  $S_5$  from the modules of the following  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -family can, for instance, not all be factored through a fixed  $A \in \mathcal{P}^\infty(\Lambda\text{-mod})$ :



(On the other hand, observe that they can be factored through the module  $A_6$  over the algebra in C.1.)

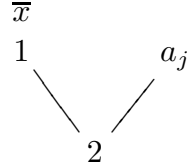
To sketch justifications for some of the claims in the preceding examples, we first spell out an obvious sufficient condition for a simple module  $S = \Lambda e / J e$  to have a  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximation. Indeed, this is the case provided that the following is true: There exist indecomposable modules  $T_1, \dots, T_m$  in  $\mathcal{P}^\infty(\Lambda\text{-mod})$  with top elements  $x_i \in T_i$  of type  $e$  such that for each indecomposable object  $X \in \mathcal{P}^\infty(\Lambda\text{-mod})$  having a top element  $x$  of type  $e$ , there exists a factor module  $X/Y$  with  $\bar{x} = x + Y \neq 0$  which can be embedded into some  $T_j$  in such a fashion that  $\bar{x}$  is mapped to  $x_j$ . If there exist  $T_1, \dots, T_m$  as stipulated, we know moreover that the indecomposable direct summands of the minimal  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximations of  $S$  are all recruited from the  $T_i$ .

**Ad Example A.1.** In arguing that  $A_1$  is the minimal  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximation of  $S_1$ , we bypass the facts that  $\text{lfdim } \Lambda = 1$  (use the method of [30], for instance), that the module  $A_1$  has finite projective dimension and satisfies the required minimality condition. To verify that  $A_1$  is a  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximation of  $S_1$ , we let  $X \in \mathcal{P}^\infty(\Lambda\text{-mod})$  be indecomposable and endowed with a top element  $x$  of type  $e_1$ . Then  $\alpha x \neq 0$ , since otherwise

the module with graph  $\begin{array}{c} 2 \\ | \\ 3 \end{array}$  would be a direct summand of  $\Omega^1(X)$ , which is impossible. If  $\gamma \alpha x \neq 0$ , then  $X \cong \Lambda x \cong \Lambda e_1$  (use indecomposability and finite projective dimension),

and  $X/\text{soc } X$  embeds into each of the modules  $\begin{array}{c} 1 \\ \diagdown \quad \diagup \\ 2 \end{array} \begin{array}{c} a_i \end{array}$  for  $i = 1, \dots, n$ . So suppose

that  $\gamma\alpha x = 0$ . In this case,  $J^2 X = 0$ , the socle of  $X$  being homogeneous of type  $e_2$ , and  $\Omega^1(X) \cong (\Lambda e_2)^r$  for some  $r$ . One infers the existence of a factor module  $X/Y$  with graph

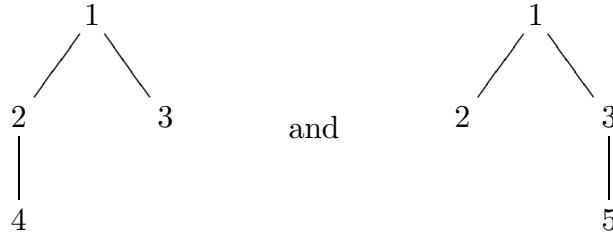


for some  $j$ .  $\square$

**Ad A.2.** To see that, also in this example, the exhibited module  $A_1$  is the minimal  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximation of  $S_1$ , observe that  $\Omega^1(A_1) = \begin{pmatrix} 2 \\ \mid \\ \delta \\ 3 \end{pmatrix}^n = (\Lambda e_2)^n$ , whence

$A_1 \in \mathcal{P}^\infty(\Lambda\text{-mod})$ . The rest of the argument is similar to the one given above.  $\square$

**Ad B.2.** Once more, we will show that  $A_1$  is a  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximation of  $S_1$  (minimality being clear then). Any indecomposable module in  $\mathcal{P}^\infty(\Lambda\text{-mod})$  containing  $\Lambda e_1$  is clearly isomorphic to  $\Lambda e_1$ , and the only proper nonzero factor modules of  $\Lambda e_1$  which embed into indecomposable modules  $X \in \mathcal{P}^\infty(\Lambda\text{-mod})$  are the direct summands of  $A_1$ , as well as

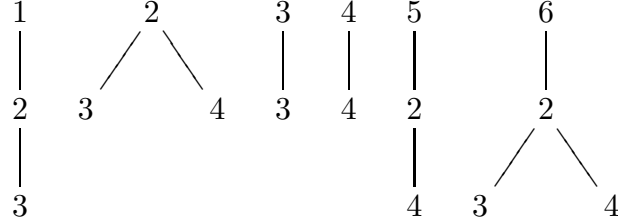


Observe that none of these factors of  $\Lambda e_1$  has a proper extension to an indecomposable module in  $\mathcal{P}^\infty(\Lambda\text{-mod})$ . Since clearly the former factors are in turn factor modules of the latter, our claim follows.  $\square$

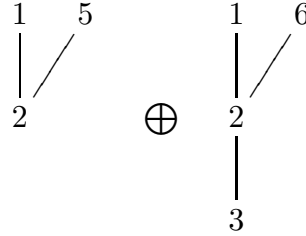
All direct summands of the minimal approximations of the various simple modules  $S = \Lambda e/Je$  exhibited above contain factor modules of  $\Lambda e$  which are minimal with respect to the property of being top-embeddable into modules of finite projective dimension. (We say that a monomorphism  $A \rightarrow B$  is a *top-embedding* if it induces a monomorphism  $A/JA \rightarrow B/JB$ .) This is generally true for the minimal  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximations of simple modules, whenever  $\Lambda$  is either left serial or a string algebra (see Sections 3 and 5), a fact which greatly facilitates resolving the existence question (always positive in case of a left serial algebra and algorithmically decidable for string algebras) and the construction of such approximations. We conclude this section with an easy example showing that this cannot be expected to hold in general. In Section 5, we will follow up with an example

demonstrating that the mentioned asset of string algebras is not shared by arbitrary special biserial algebras either.

**Example D.** Let  $\Lambda = KT/I$  be the monomial relation algebra with the following indecomposable projective left modules:



Then  $\mathcal{P}^\infty(\Lambda\text{-mod})$  is contravariantly finite, the minimal  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximation  $A_1$  of  $S_1$  having the following graph:



Note that the second summand of  $A_1$  does not contain a proper factor module of  $\Lambda e_1$ , whereas  $\Lambda e_1 / \text{soc}(\Lambda e_1)$  is the (unique in this case) minimal factor module of  $\Lambda e_1$  which can be top-embedded into an object of  $\mathcal{P}^\infty(\Lambda\text{-mod})$ .

### 3. HOMOLOGICAL IMPORTANCE OF CONTRAVARIANT FINITENESS AND A MODEL APPLICATION

As surfaced in [31], [33] and [48], non-finitely generated modules of finite projective dimension may display structural phenomena which are completely different from those encountered in finitely generated modules of finite projective dimension. (We labeled them ‘domino effects’ in [31].) In particular, there may be objects in  $\mathcal{P}^\infty(\Lambda\text{-Mod})$  whose projective dimension exceeds  $\text{l fin dim } \Lambda$  by any predetermined positive number, even when  $\text{l fin dim } \Lambda = 1$  [48]. Moreover, the left cyclic finitistic dimension  $\text{l cyc fin dim } \Lambda$ , i.e., the supremum of those projective dimensions which are attained on the cyclic modules in  $\mathcal{P}^\infty(\Lambda\text{-mod})$ , may be strictly smaller than  $\text{l fin dim } \Lambda$ . In fact, for each natural number  $n$ , there exists a finite dimensional algebra  $\Lambda_n$  such that  $\text{l fin dim } \Lambda_n$  is not attained on any  $n$ -generated module (see [33]). However, in case  $\mathcal{P}^\infty(\Lambda\text{-mod})$  is contravariantly finite, all of the left finitistic dimensions of  $\Lambda$  coincide, and the objects of the big category  $\mathcal{P}^\infty(\Lambda\text{-Mod})$  are as well understood as those of the small  $\mathcal{P}^\infty(\Lambda\text{-mod})$ .

The following notation will be convenient: Given objects  $A_1, \dots, A_n$  in  $\Lambda\text{-mod}$ , let  $\text{filt}(A_1, \dots, A_n)$  be the full subcategory of  $\Lambda\text{-mod}$  the objects of which are those modules which have filtrations with consecutive factors among  $A_1, \dots, A_n$ . More precisely,



$X$  belongs to  $\text{filt}(A_1, \dots, A_n)$  if and only if there exists a chain  $X = X_0 \supseteq X_1 \supseteq \dots \supseteq X_m = 0$  such that each of the factors  $X_i/X_{i+1}$  is isomorphic to some  $A_{j(i)}$ . Moreover,  $\overrightarrow{\text{filt}}(A_1, \dots, A_n)$  will stand for the closure of  $\text{filt}(A_1, \dots, A_n)$  under direct limits in  $\Lambda\text{-Mod}$ .

Concerning the structure of the *finitely generated* modules of finite projective dimension in case  $\mathcal{A} = \mathcal{P}^\infty(\Lambda\text{-mod})$  is contravariantly finite, Auslander and Reiten proved the following result in the more general context of an arbitrary resolving subcategory  $\mathcal{A}$ .

**Theorem 2.** [3] *Suppose that  $\mathcal{A}$  is a resolving contravariantly finite subcategory of  $\Lambda\text{-mod}$ , and that  $A_1, \dots, A_n$  are the minimal  $\mathcal{A}$ -approximations of the simple left  $\Lambda$ -modules. Then a module  $X$  belongs to  $\mathcal{A}$  if and only if  $X$  is a direct summand of an object in  $\text{filt}(A_1, \dots, A_n)$ .  $\square$*

Of course, this theorem, applied to  $\mathcal{A} = \mathcal{P}^\infty(\Lambda\text{-mod})$ , yields the following consequence:

**Corollary 3.** [3] *If  $\mathcal{P}^\infty(\Lambda\text{-mod})$  is contravariantly finite and  $A_1, \dots, A_n$  are the minimal  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximations of the simple left  $\Lambda$ -modules, then*

$$\text{l fin dim } \Lambda = \text{l cyc fin dim } \Lambda = \max\{\text{p dim } A_1, \dots, \text{p dim } A_n\}. \quad \square$$

Due to the fact that contravariant finiteness of  $\mathcal{P}^\infty(\Lambda\text{-mod})$  places demands only on the finitely generated modules, the strong impact which this condition has on *non-finitely generated* modules may come as a surprise. In fact, the structure theory for objects in  $\mathcal{P}^\infty(\Lambda\text{-mod})$  extends smoothly to  $\mathcal{P}^\infty(\Lambda\text{-Mod})$ .

**Theorem 4.** [35] *Again suppose that  $\mathcal{P}^\infty(\Lambda\text{-mod})$  is contravariantly finite, and let  $A_1, \dots, A_n$  be as in the corollary. Then*

$$\mathcal{P}^\infty(\Lambda\text{-Mod}) = \overrightarrow{\text{filt}}(A_1, \dots, A_n),$$

and, in particular,

$$\text{l Fin dim } \Lambda = \text{l fin dim } \Lambda = \max\{\text{p dim } A_1, \dots, \text{p dim } A_n\}. \quad \square$$

Thus, contravariant finiteness of  $\mathcal{P}^\infty(\Lambda\text{-mod})$  resolves the notorious quandary of locating objects in  $\Lambda\text{-Mod}$  on which  $\text{l Fin dim } \Lambda$  is attained; this search may be a very difficult task, even when finiteness of  $\text{l Fin dim } \Lambda$  is guaranteed in advance. The helpfulness of the above theory will be displayed to full advantage in our examples.

### Examples of Section 2 revisited.

For the moment, we will only determine the finitistic dimensions of those algebras displayed which give rise to contravariantly finite categories  $\mathcal{P}^\infty(-)$ . By the preceding discussion, the objects of  $\Lambda\text{-Mod}$  having finite projective dimension are precisely the direct limits of the objects in  $\text{filt}(A_1, \dots, A_n)$  with the  $A_i$  as shown in Section 2.

**Ad A.1.** It is clear that  $\text{p dim } A_1 = \text{p dim } A_{a_i} = 1$  for  $i = 1, \dots, n$ , whereas  $\text{p dim } A_2 = \text{p dim } A_3 = 0$ . Hence,  $\text{l Fin dim } \Lambda = \text{l fin dim } \Lambda = 1$ .

**Ad A.2.** In this example, the minimal  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximation of each simple  $S_{a_i}$  centered in the vertex  $a_i$  coincides with  $S_{a_i}$  and has projective dimension 1, as does the minimal approximation of  $S_1$ . The minimal approximations of  $S_2$  and  $S_3$  are again identical with their projective covers. So, once more,  $\text{l Fin dim } \Lambda = \text{l fin dim } \Lambda = 1$ .

**Ad B.2.** Here  $A_1, A_4, A_5, A_6$  are projective, while  $\text{p dim } A_2 = \text{p dim } A_3 = 1$ , and we obtain the same conclusion as before.  $\square$

We will briefly digress from our main line of thought for another corollary of the preceding theorem. The existence theorem for internal almost split sequences [4] which we quoted at the outset can be strengthened for the category  $\mathcal{A} = \mathcal{P}^\infty(\Lambda\text{-mod})$  as follows.

**Corollary 5.** *If  $\mathcal{P}^\infty(\Lambda\text{-mod})$  is contravariantly finite, then  $\mathcal{P}^\infty(\Lambda\text{-mod})$  is also covariantly finite and thus has almost split sequences.*

*Proof.* By a result of Crawley-Boevey [15], it suffices to show that arbitrary direct products of objects in the category  $\mathcal{P}^\infty(\Lambda\text{-mod})$  belong to its closure  $\overrightarrow{\mathcal{P}^\infty}(\Lambda\text{-mod})$  under direct limits in  $\Lambda\text{-Mod}$ . But in view of the theorem, contravariant finiteness of  $\mathcal{P}^\infty(\Lambda\text{-mod})$  entails the equality  $\mathcal{P}^\infty(\Lambda\text{-Mod}) = \overrightarrow{\mathcal{P}^\infty}(\Lambda\text{-mod})$  and, in view of the finiteness of  $\text{l Fin dim } \Lambda$ , this guarantees closedness of this latter subcategory under direct products.  $\square$

As a class of examples of algebras  $\Lambda$  with very rich and complex module categories, for which the above program of zeroing in on the structure of the objects in  $\mathcal{P}^\infty(\Lambda\text{-Mod})$  works to perfection, we will present the left serial algebras. Recall that a split algebra  $\Lambda = K\Gamma/I$  is called *left serial* in case no more than one arrow leaves any given vertex of  $\Gamma$ ; equivalently, this means that the indecomposable projective left  $\Lambda$ -modules are all uniserial. To describe the minimal  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximations of the simple modules in this situation, we require the following definition.

**Definition.** Suppose that  $T_1, \dots, T_m$  is a sequence of nonzero uniserial left  $\Lambda$ -modules, and let  $p_i$  be a *mast* of  $T_i$ , namely a path in  $K\Gamma$  of maximal length with  $p_i T_i \neq 0$ . A left  $\Lambda$ -module  $T$  is called a *saguaro* on  $(p_1, \dots, p_m)$  if

(i)  $T \cong (\bigoplus_{1 \leq i \leq m} T_i)/U$ , where  $U \subseteq \bigoplus_{1 \leq i \leq m} JT_i$  is generated by a sequence of elements of the form  $q_i t_i - q'_{i+1} t_{i+1}$ ,  $1 \leq i \leq m-1$ , where  $t_i \in T_i$  are suitable top elements and  $q_i, q'_i$  are right subpaths of the masts  $p_i$  such that  $q_i t_i \neq 0$ , and  $q'_{i+1} t_{i+1} \neq 0$ ; moreover, we require that

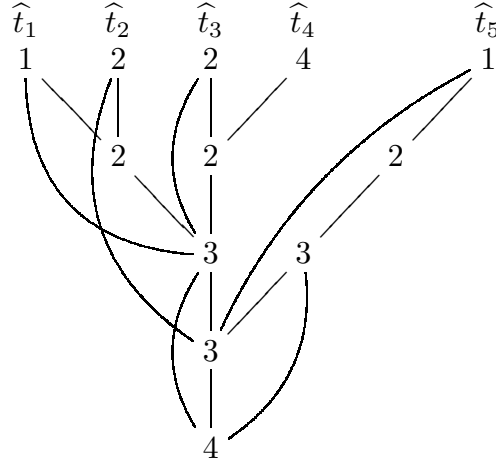
(ii) each  $T_j$  embeds canonically into  $T$  via

$$T_j \xrightarrow{\text{can}} \left( \bigoplus_{1 \leq i \leq m} T_i \right) / U \cong T.$$

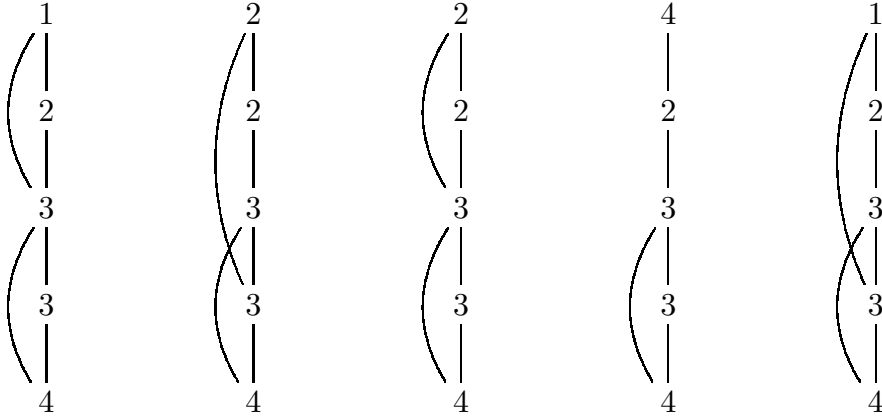
The uniserial modules  $T_i$  are called the *trunks* of  $T$ .

We will identify  $T$  with  $(\bigoplus_{1 \leq i \leq m} T_i)/U$ . To avoid ambiguities, we will denote the canonical images of the trunks  $T_i$  inside  $T$  by  $\hat{T}_i$  and the canonical images of the top elements  $t_i$  by  $\hat{t}_i$ . Any such sequence  $(\hat{t}_1, \dots, \hat{t}_m)$  will be called a *canonical sequence of top elements* for  $T$ .

Note that saguaros are particularly amenable to graphing, the shape of their graphs explaining their name (they share shape and name with a cactus found in the Sonoran desert, *Cereus giganteus*). In fact, the definition forces them to be glued together in a very straightforward fashion from their uniserial trunks: Layered and labeled graphs relative to a canonical sequence of top elements always exist (not only over left serial algebras), and are built on the pattern illustrated below.



Here  $T = (\bigoplus_{i=1}^5 T_i)/U$ , where the trunks  $T_i = \Lambda t_i$  of  $T$  have graphs



relative to the top elements  $t_i$ .

Of course, over left serial algebras, the graphs of the uniserial left modules are necessarily edge paths (in particular, uniserials have unique masts), so that the graphs of saguaros simplify to trees. It turns out that, in this situation, all minimal  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximations of simple modules are recruited from the class of saguaros with simple socles. In fact, these approximating saguaros can be pinned down explicitly and even constructed algorithmically on the basis of quiver and relations.

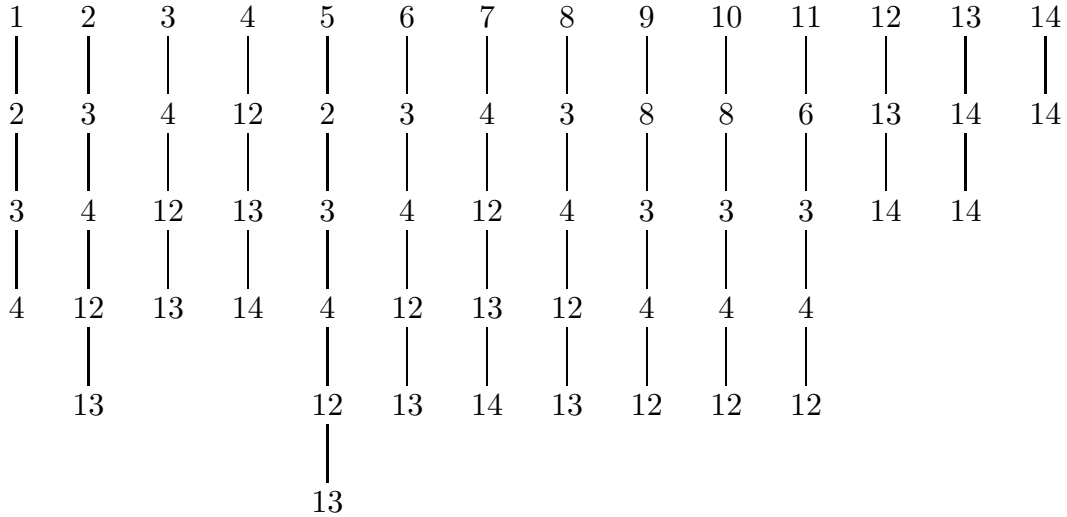
**Theorem 6.** [12] *Suppose that  $\Lambda = K\Gamma/I$  is a left serial algebra. Then  $\mathcal{P}^\infty(\Lambda\text{-mod})$  is contravariantly finite, and the minimal  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximations of the simple left  $\Lambda$ -modules are saguaros with simple socles.*

More precisely, the minimal  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximation of a simple left  $\Lambda$ -module  $S = \Lambda e/Je$  can be described as follows: If  $\Lambda e/C$  is the (unique) minimal nonzero factor module of  $\Lambda e$  which has finite projective dimension, there is a unique saguaro  $A(S)$  of maximal length in  $\mathcal{P}^\infty(\Lambda\text{-mod})$  such that  $\Lambda e/C$  is a trunk of  $A(S)$  and  $\text{soc } A(S)$  is simple. Moreover, the canonical epimorphisms  $A(S) \rightarrow S$ , which map  $\Lambda e/C$  onto  $S$  and send the other trunks of  $A(S)$  to zero, are minimal  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximations.  $\square$

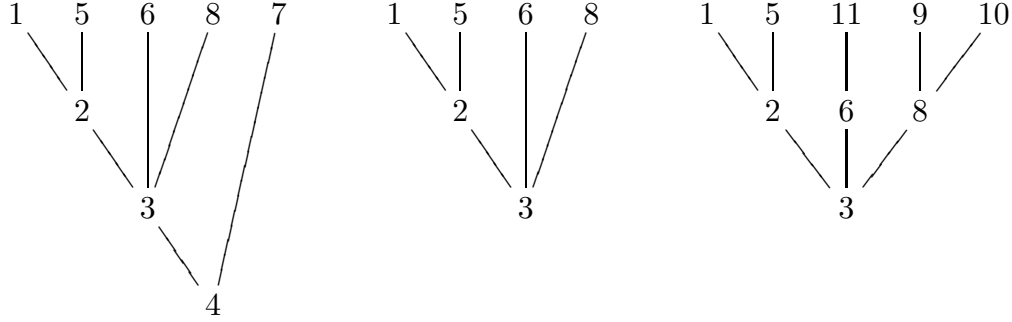
To refer back to the concluding remark of Section 2: In the setting of the theorem,  $\Lambda e/C$  actually coincides with the nonzero factor module of  $\Lambda e$  which is minimal with respect to top-embeddability into a module of finite projective dimension.

Actually, not only is  $\mathcal{P}^\infty(\Lambda\text{-mod})$  always contravariantly finite in the left serial case, but so are the categories  $\mathcal{P}^{(d)} = \mathcal{P}^{(d)}(\Lambda\text{-mod})$  consisting of the finitely generated left  $\Lambda$ -modules of projective dimensions at most  $d$ . Moreover, the minimal  $\mathcal{P}^{(d)}$ -approximations of the simples are again saguaros, and the sequences of these saguaros for  $1 \leq d \leq \text{fin dim } \Lambda$  record the homological properties of  $\Lambda$  with high precision, the case  $d = \text{fin dim } \Lambda$  leading back to  $\mathcal{P}^\infty(\Lambda\text{-mod})$ .

**Example E.** Let  $\Lambda$  be a left serial algebra whose indecomposable projective modules are represented by the following graphs.



The evolution of the  $\mathcal{P}^{(d)}$ -approximations of the simple left  $\Lambda$ -module  $S_1$  is graphically represented below. We exhibit the minimal  $\mathcal{P}^{(1)}$ -,  $\mathcal{P}^{(2)}$ -,  $\mathcal{P}^{(3)}$ -approximations of  $S_1$  from left to right; the last coincides with the minimal  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximation, since the left finitistic dimension of  $\Lambda$  is 3.



#### 4. PHANTOMS. DEFINITIONS, EXISTENCE, AND BASIC PROPERTIES

The objects discussed in this section were introduced by Happel and the author in [29]. As mentioned in the introduction, their purpose is twofold: In the first place, they serve as indicators as to whether or not a given subcategory  $\mathcal{A} \subseteq \Lambda\text{-mod}$  is contravariantly finite. Their second role is that of retaining the kind of information which is stored in minimal  $\mathcal{A}$ -approximations whenever they exist, within potentially infinite dimensional frames; this role is played most satisfactorily by the ‘effective’ phantoms.

Since the concept of a phantom is possibly not easily translated into an intuitive picture, we break the definitions into several parts, and add a fairly extensive discussion at each step. The subsequent detailed analysis of the examples of Section 2 should also help the reader to see that we are dealing with objects that arise very naturally when the relations of a module  $X$  are being compared with those of the objects in a subcategory of  $\Lambda\text{-mod}$ .

**Definition, Part I.** (Relative approximations) Let  $\mathcal{A}$  be a full subcategory of  $\Lambda\text{-mod}$ , and  $\mathcal{C}$  a subcategory of  $\mathcal{A}$ . Moreover, let  $X \in \Lambda\text{-mod}$ .

A  $\mathcal{C}$ -approximation of  $X$  inside  $\mathcal{A}$  is a homomorphism  $f : A \rightarrow X$  with  $A \in \mathcal{A}$  such that each map in  $\text{Hom}_\Lambda(C, X)$  with  $C \in \mathcal{C}$  factors through  $f$ . Again, we will loosely refer to the object  $A$  as a  $\mathcal{C}$ -approximation of  $X$  inside  $\mathcal{A}$ .

Clearly, whenever  $X$  has a minimal  $\mathcal{A}$ -approximation,  $A_0$  say, the classical  $\mathcal{A}$ -approximations of  $X$  are precisely the  $\{A_0\}$ -approximations of  $X$  inside  $\mathcal{A}$ . In particular, we then obtain  $A_0$  as a  $\mathcal{C}$ -approximation of  $X$  inside  $\mathcal{A}$ , where  $\mathcal{C}$  is a finite subcategory of  $\mathcal{A}$ . Moreover,  $A_0$  is a direct summand of any  $\{A_0\}$ -approximation of  $X$  inside  $\mathcal{A}$ , and so, a fortiori, is a subfactor of any such approximation. On the other hand, given a finite subcategory  $\mathcal{C} \subseteq \mathcal{A}$ , the module  $X$  will have  $\mathcal{C}$ -approximations inside  $\mathcal{A}$  provided that we require  $\mathcal{A}$  to be closed under finite direct sums: Just sum up a sufficient number of copies of the objects in  $\mathcal{C}$ . In other words, approximations relative to finite classes are always available, also in case there are no classical  $\mathcal{A}$ -approximations of  $X$ .

**Definition, Part II.** (Phantoms) Retain the notation of Part I, and suppose, in addition, that  $\mathcal{A}$  is closed under finite direct sums.

A finitely generated module  $H \in \Lambda\text{-mod}$  is an  $\mathcal{A}$ -phantom of  $X$  in case

(\*) there is a finite subcategory  $\mathcal{A}' \subseteq \mathcal{A}$  such that  $H$  occurs as a subfactor of *every*  $\mathcal{A}'$ -approximation of  $X$  inside  $\mathcal{A}$ .

More generally, an arbitrary module  $H \in \Lambda\text{-Mod}$  will be called an  $\mathcal{A}$ -phantom of  $X$  if each of its finitely generated submodules satisfies (\*). Of course, the choices of the finite subclasses  $\mathcal{A}' \subseteq \mathcal{A}$  will vary with the finitely generated submodules  $H'$  of  $H$ .

The class of all  $\mathcal{A}$ -phantoms of  $X$  is clearly closed under subfactors, and consequently is closed under direct limits of direct systems as well. In fact, a module  $H \in \Lambda\text{-Mod}$  is an  $\mathcal{A}$ -phantom of  $X$  if and only if  $H$  is the direct limit of a direct system of finitely generated  $\mathcal{A}$ -phantoms of  $X$ . This may make the class of  $\mathcal{A}$ -phantoms enormous: Indeed, the class of all  $\mathcal{A}$ -phantoms of a simple module  $S$  may encompass the entire class of indecomposables  $H$  in  $\Lambda\text{-Mod}$  with  $S \hookrightarrow H/JH$ . This slack in the definition of phantoms has the advantage of facilitating their construction. Often, the particular structure of phantoms is irrelevant – their sheer size is enough to signal non-existence of traditional  $\mathcal{A}$ -approximations of  $X$ . In fact, if  $X$  has an  $\mathcal{A}$ -approximation then the class of  $\mathcal{A}$ -phantoms of  $X$  coincides with the set of subfactors of the minimal such approximation. Consequently, the existence of phantoms of  $X$  of unbounded lengths is enough to guarantee non-existence of traditional  $\mathcal{A}$ -approximations. Of course, in terms of encapsulating further information, the usefulness of phantoms so generously defined is moderate. Hence, we single out a subclass of phantoms which are more strongly tied to the category  $\mathcal{A}$  and carry a full complement of information on how a given subcategory  $\mathcal{C} \subseteq \mathcal{A}$  relates to  $X$ .

**Definition, Part III.** (Effective phantoms) Keep the notation of Part II, and denote by  $\overrightarrow{\mathcal{A}}$  the closure of  $\mathcal{A}$  under direct limits (of direct systems) in  $\Lambda\text{-Mod}$ . Moreover, fix a subcategory  $\mathcal{C} \subseteq \mathcal{A}$ .

An  $\mathcal{A}$ -phantom  $H \in \overrightarrow{\mathcal{A}}$  is called *effective relative to  $\mathcal{C}$*  if there exists a homomorphism  $h : H \rightarrow X$  with the property that each map in  $\text{Hom}_\Lambda(C, X)$  with  $C \in \mathcal{C}$  factors through  $h$ .

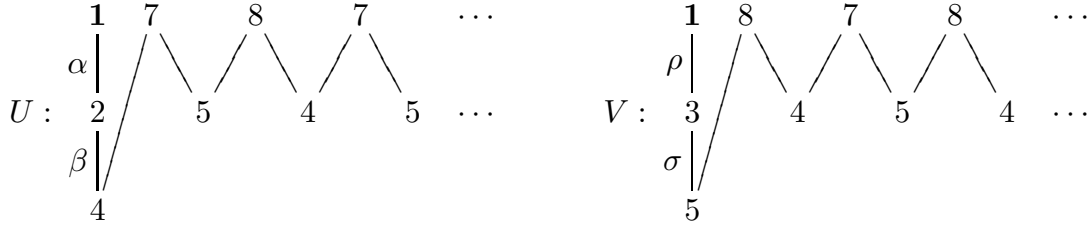
In a self-explanatory extension of the term ‘approximation’ to subcategories of  $\Lambda\text{-Mod}$ , the effectiveness condition thus calls for  $h : H \rightarrow X$  to be a  $\mathcal{C}$ -approximation of  $X$  inside  $\overrightarrow{\mathcal{A}}$ . In case  $X$  has a traditional  $\mathcal{A}$ -approximation, the minimal such approximation is clearly the only effective  $\mathcal{A}$ -phantom of  $X$  relative to  $\mathcal{A}$ . Otherwise, existence of interesting phantoms, effective or not, is not immediately clear, but is guaranteed by the following result.

**Theorem 7.** [29] *Let  $\mathcal{A} \subseteq \Lambda\text{-mod}$  be a full subcategory which is closed under finite direct sums. For  $X \in \Lambda\text{-mod}$ , the following conditions are equivalent:*

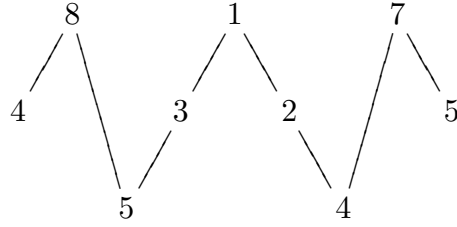
- (1)  *$X$  fails to have an  $\mathcal{A}$ -approximation.*
- (2)  *$X$  has  $\mathcal{A}$ -phantoms of infinite  $K$ -dimension.*
- (3) *There exist countable subclasses  $\mathcal{C} \subseteq \mathcal{A}$  such that  $X$  has infinite dimensional  $\mathcal{A}$ -phantoms which are effective relative to  $\mathcal{C}$ .  $\square$*

Next we discuss our ‘test examples’ in light of the new concepts.

**Ad B.1.** Here are two infinite dimensional  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -phantoms of  $S_1$ :

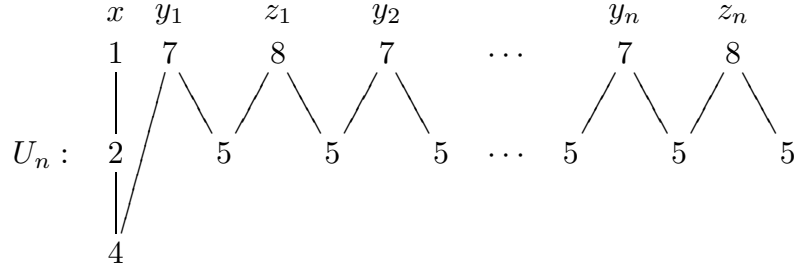


Note that  $U \oplus V$  is in turn a  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -phantom of  $S_1$ , since there is no object in  $\mathcal{P}^\infty(\Lambda\text{-Mod})$  having a graph with subgraph



In fact, the phantom  $U \oplus V$  is effective relative to the class  $\mathcal{C} \subseteq \mathcal{P}^\infty(\Lambda\text{-mod})$  exhibited in B.1.

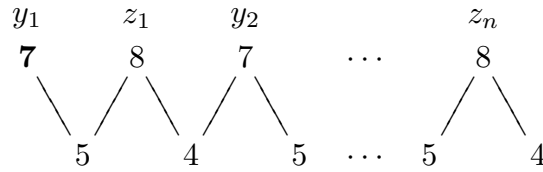
We will justify only that  $U$  is a  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -phantom of  $S_1$ . Consider the class of objects



in  $\mathcal{P}^\infty(\Lambda\text{-mod})$ , and observe that  $U = \varinjlim U_n$ . Hence, it suffices to show that, for each  $n \in \mathbb{N}$ , the module  $U_n$  is a submodule of each  $\{U_n\}$ -approximation of  $S_1$  inside  $\mathcal{P}^\infty(\Lambda\text{-mod})$ . To see this, fix  $n$ , and let  $A$  be any  $\{U_n\}$ -approximation of  $S_1$  inside  $\mathcal{P}^\infty(\Lambda\text{-mod})$ ; say  $f : A \rightarrow S_1$  has the factorization property of the definition, and  $g \in \text{Hom}(U_n, A)$  factors the canonical epimorphism  $U_n \rightarrow S_1$ . Let  $a = g(x)$ , where  $x$  is the unique top element of type  $e_1$  of  $U_n$ . Then  $a$  is a top element of  $A$ . Due to the fact that finiteness of the projective dimension of  $A$  entails either  $\beta\alpha a \neq 0$  or  $\sigma\rho a \neq 0$ , our factorization requirement forces  $\beta\alpha a$  to be nonzero. Consequently,  $g(\beta\alpha x) = \beta\alpha a$ . If the arrows  $7 \rightarrow 4$  and  $7 \rightarrow 5$  are named  $\chi$  and  $\psi$ , respectively, we deduce  $\beta\alpha a = f(\chi y_1) \neq 0$ , and hence  $f(y_1) \neq 0$ . The element  $b_1 = f(y_1)$  is a top element of  $A$  (necessarily of type  $e_7$ ), since 7 is a source of  $\Gamma$ . To prevent the syzygy  $\Omega^1(A)$  from having a summand  $S_5$  (the latter being incompatible with finite projective dimension), we require  $0 \neq \psi(b_1) = \mu f(z_1)$ , where  $\mu$  is the arrow

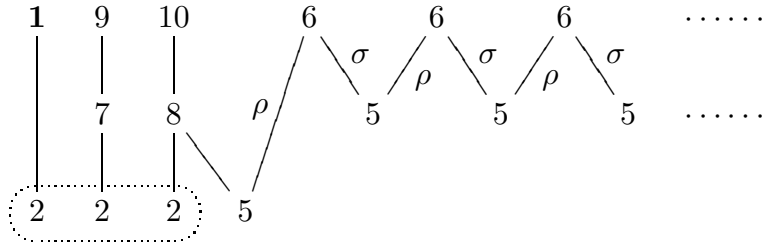
$8 \rightarrow 5$  and  $z_1$  is as indicated in the above figure. Setting  $c_1 = f(z_1)$  and iterating this type of argument, we see that  $0 \neq \nu c_1 = f(\chi y_2)$ , where  $\nu$  is the arrow  $8 \rightarrow 4$ . Set  $b_2 = f(y_2)$ . It is readily checked that the top elements  $b_1, b_2$  are  $K$ -linearly independent modulo  $JA$ , and an obvious induction on  $m \leq n$  gives us sequences of top elements,  $b_1, \dots, b_n$  of type  $e_7$ , and  $c_1, \dots, c_n$  of type  $e_8$  in  $A$ , both series being  $K$ -linearly independent modulo  $JA$ . It is now straightforward to deduce that the submodule of  $A$  generated by  $a$  and the  $b_i, c_i$  has the same graph as  $U_n$ . But this graph clearly determines the corresponding module up to isomorphism, which completes the argument.

Each module from the subclass  $\mathcal{D} = \{D_n \mid n \in \mathbb{N}\}$  of  $\mathcal{P}^\infty(\Lambda\text{-mod})$ , with  $D_n$  determined by the graph



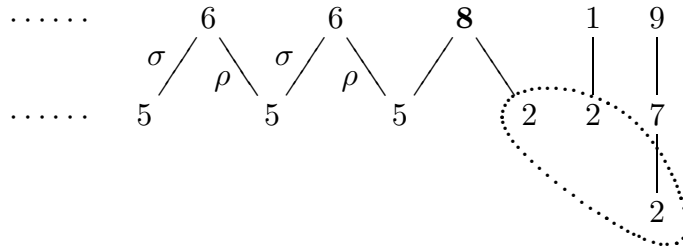
is a  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -phantom of  $S_7$ , and consequently  $\varinjlim D_n$  provides us with an infinite dimensional  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -phantom of  $S_7$  which, moreover, is effective relative to the class  $\mathcal{D}$ . The simple module  $S_8$  behaves similarly.

**Ad C.2.** An infinite dimensional  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -phantom of  $S_1$  is determined (uniquely, up to isomorphism) by the graph



This phantom is effective relative to the class of modules in  $\mathcal{P}^\infty(\Lambda\text{-mod})$  obtained by chopping suitable ‘infinite tails’ off the given graph.

An example of an infinite dimensional  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -phantom of  $S_8$ , finally, is



This concludes our discussion of Examples A,B,C. The following criterion from [29] is hovering in the background of most of the infinite dimensional phantoms displayed so far.

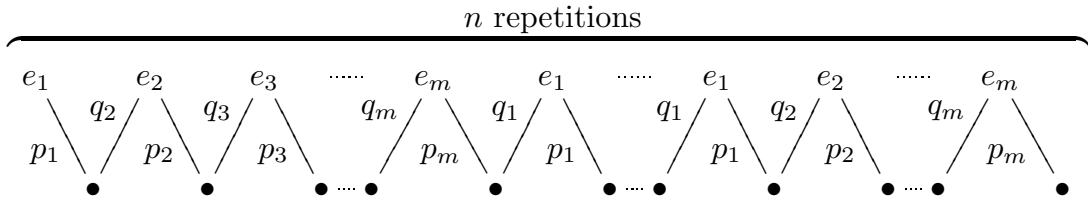


**Criterion for failure of contravariant finiteness of  $\mathcal{A}$ .**

Let  $\Lambda = K\Gamma/I$  be a split finite dimensional algebra, and, once more, suppose that  $\mathcal{A} \subseteq \Lambda\text{-mod}$  is closed under finite direct sums. The simple module  $S = \Lambda e_1 / J e_1$  centered in the vertex  $e_1$  fails to have an  $\mathcal{A}$ -approximation in case the following holds:

The vertex  $e_1$  can be supplemented to a sequence  $e_1, \dots, e_m$  of distinct vertices of  $\Gamma$ , together with sequences  $p_1, \dots, p_m, q_1, \dots, q_m$  in  $J$ , where  $p_i = p_i e_i$  and  $q_i = q_i e_i$  are such that conditions (1) and (2) below are satisfied:

(1) For each  $n \in \mathbb{N}$ , there exists a module  $M_n \in \mathcal{A}$  having a graph that contains a subgraph of the form



(2) Given any object  $A$  in  $\mathcal{A}$ , the top elements of  $A$  of type  $e_1$  are not annihilated by  $p_1$ , and

$$(p_i a = q_{i+1} b \neq 0 \implies p_{i+1} b \neq 0)$$

for  $a, b \in A$  and  $1 \leq i \leq m$ ; here  $p_{m+1} = p_1$  and  $q_{m+1} = q_1$ .  $\square$

In fact, the hypotheses of the criterion yield an infinite dimensional factor module of  $\varinjlim M_n$  which is an  $\mathcal{A}$ -phantom of  $S$ ; it has a graph containing the ‘infinite extension to the right’ of the graph in condition (1) as a subgraph.

**5. PHANTOMS OVER STRING ALGEBRAS**

In this section, we present a preview of ongoing joint work with S. O. Smalø.

A special class of string algebras  $\Lambda_{m,n}$ , for  $m, n \geq 2$  – those on the quiver

$$\alpha \circlearrowleft \bullet \circlearrowright \beta$$

and subject to the relations  $\alpha\beta = \beta\alpha = 0$  and  $\alpha^m = \beta^n = 0$  – was first singled out by Gelfand and Ponomarev in the late 1960’s, as being intimately related to the representation theory of the Lorentz group [28]; in fact, classifying the finitely generated indecomposable modules over the algebras  $\Lambda_{m,n}$  amounts to a classification of the Harish-Chandra modules over the Lorentz group. Taking this route, Gelfand and Ponomarev gave a hands-on structural description of the finitely generated indecomposable objects in  $\Lambda_{m,n}\text{-mod}$ . In particular, their findings show that over an algebraically closed base field the algebras  $\Lambda_{m,n}$  are tame. Actually, this work was preceded by an investigation of Szekeres [49] into the structure of the finitely generated modules over certain factor rings of  $\mathbb{Z}[x]$ , which anticipates many of the ideas re-encountered in later work on biserial algebras. In the

mid-seventies, Gabriel presented a categorical reinterpretation of the Gelfand-Ponomarev approach (see [24]), which in turn caused Ringel to recognize that these methods are applicable in a far wider context: In a first round of generalizations, he used them to describe the finite dimensional indecomposable representations of the dihedral 2-groups in characteristic 2 [43]; this work appeared in tandem with a paper of Bondarenko containing roughly the same information [11]. Next, Donovan and Freislich applied these methods to the ‘biserial’ algebras introduced below [17]. Clearly, the algebras  $\Lambda_{m,n}$  considered by Gelfand and Ponomarev belong to the subclass of ‘special biserial’ algebras; in fact, they are even ‘string algebras’.

**Definitions.** (see [51] and [13]) (1)  $\Lambda$  is called *biserial* if each indecomposable projective left or right  $\Lambda$ -module  $P$  has the following property:  $\text{rad } P = U + V$ , where  $U, V$  are uniserial (possibly trivial) with  $U \cap V$  either zero or simple.

(2)  $\Lambda$  is *special biserial* provided  $\Lambda$  is of the form  $K\Gamma/I$  such that

- Given any vertex  $e$  of  $\Gamma$ , there are at most two arrows entering  $e$  and at most two arrows leaving  $e$ , and
- Given any arrow  $\alpha$  of  $\Gamma$ , there exists at most one arrow  $\beta$  such that  $\beta\alpha$  does not belong to  $I$ , and at most one arrow  $\gamma$  such that  $\alpha\gamma$  does not belong to  $I$ .

Moreover,  $\Lambda$  is a *string algebra* if, in addition,  $\Lambda$  is a monomial relation algebra, meaning that  $I$  can be generated by certain paths in  $\Gamma$ .

Clearly, special biserial algebras are biserial. All finite dimensional biserial algebras over algebraically closed fields are known to be tame: the special biserial case was completed by Wald and Waschbüsch in [51], while the general biserial situation was settled much later by Crawley-Boevey [16] on the basis of an alternate description of biserial algebras due to him and Vila-Freyer [50] and a remarkable result of Geiss [25] (saying that algebras with tame degenerations are always tame).

All the while, special biserial algebras have continued to provide challenges which, in spite of the availability of a highly explicit classification of the indecomposables, were far from resolvable at a glance. We mention only a few such lines, together with a selection of references, not aiming at completeness:

Let  $\Lambda$  be biserial.

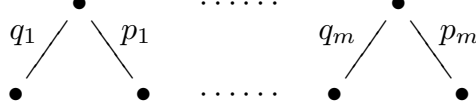
- When does  $\Lambda$  have finite representation type? (See [47].)
- What does the Auslander-Reiten quiver of  $\Lambda$  look like? (See, e.g., [13], [21], and [26].)
- Characterize the auto-equivalences of the category  $\Lambda\text{-mod}$ . (See [9] and [10].)
- Describe the generic modules over string algebras and the maps among them. (See [45]).

New sources of symmetric biserial algebras can be found in [46].

Here we will provide an overview of a complete, constructive solution to the problem as to which string algebras  $\Lambda$  have the property that  $\mathcal{P}^\infty(\Lambda\text{-mod})$  is contravariantly finite. Our proof of the answer – not given here – makes full use of the description of the indecomposable objects in  $\Lambda\text{-mod}$ , as reviewed graphically below.

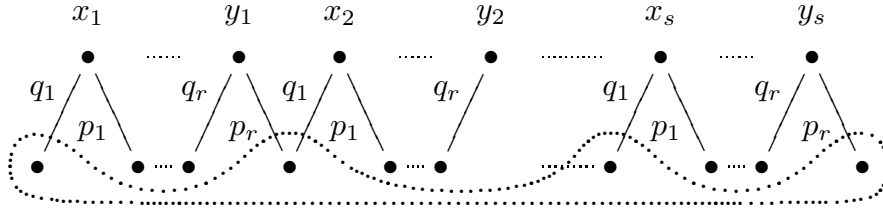
**Theorem 8.** (Its evolution can be traced in [28], [43], [17], [51]) *Let  $\Lambda = K\Gamma/I$  be a special biserial algebra over an algebraically closed field  $K$ . Then each indecomposable*

object in  $\Lambda\text{-mod}$  is either a band module or a string module. Here the string modules are those with graphs of the form



where the  $p_i, q_i$  are paths in  $K\Gamma \setminus I$ , with  $q_1$  and  $p_m$  possibly trivial, such that  $\text{firstarrow}(q_i) \neq \text{firstarrow}(p_i)$  for  $1 \leq i \leq m$ , and  $\text{lastarrow}(p_i) \neq \text{lastarrow}(q_{i+1})$  for  $1 \leq i \leq m-1$ . (Note that some of these conditions may be void, namely in case the relevant paths are trivial.)

The band modules are characterized by their graphs, paired with irreducible vector space automorphisms, as follows: The pertinent graphs are of the form



where  $p_i, q_i$  are nontrivial paths in  $K\Gamma \setminus I$  with  $\text{firstarrow}(q_i) \neq \text{firstarrow}(p_i)$  for all  $i$  and  $\text{lastarrow}(p_i) \neq \text{lastarrow}(q_{i+1})$  for  $i < r$ , and also  $\text{lastarrow}(p_r) \neq \text{lastarrow}(q_1)$ . The nature of the dotted pool is specified by the dependence relation

$$p_r y = \sum_{i=1}^s k_i q_1 x_i,$$

where  $\begin{bmatrix} 0 & \cdots & 0 & k_1 \\ 1 & \cdots & 0 & k_2 \\ & \ddots & & \vdots \\ 0 & \cdots & 1 & k_s \end{bmatrix}$  is the Frobenius companion matrix of an irreducible automorphism of  $K^s$ .

Moreover, all modules having one of the above descriptions are indecomposable.  $\square$

This classification can be completed with a suitable uniqueness statement which does not impinge on our present results. For our main theorem, string modules will be of particular relevance. A *generalized string module* will be a module  $X \in \Lambda\text{-Mod}$  which arises as a direct limit of a countable directed system of string modules, each embedded into its successor, i.e., any module  $X$  having a graph of one of the following forms



such that each finite segment is the graph of a string module. (Here the dotted edges may but need not appear, depending on whether the graph is one-/two-sided infinite or finite.)

These modules were also considered by Ringel in [44] as “modules associated with  $\mathbb{N}$ -words of  $\mathbb{Z}$ -words” over the alphabet  $\Gamma_0 \cup \Gamma_0^{-1}$ , where  $\Gamma_0$  is the vertex set of the quiver  $\Gamma$  of  $\Lambda$ . It should be self-explanatory what we mean by a *left and right periodic* generalized string module; we merely emphasize that we allow the “left” and “right” periods to differ and that we regard termination as a period.

We are now in a position to state the main new result of this section.

**Theorem 9.** [36] *Let  $\Lambda = K\Gamma/I$  be a finite dimensional string algebra. For any simple  $S \in \Lambda\text{-mod}$ , there exists a generalized string module  $H = H(S)$  which is uniquely determined by  $\Gamma$  and  $I$ , together with a canonical homomorphism  $f : H \rightarrow S$ , having the following properties:*

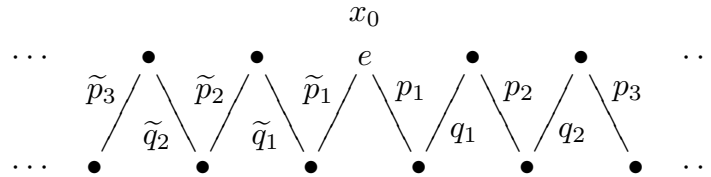
(I) *The following statements are equivalent:*

- (i)  $\dim_K H < \infty$ ;
- (ii)  $S$  has a classical  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximation;
- (iii)  $f : H \rightarrow S$  is the minimal  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximation of  $S$ .

(II)  *$H$  always belongs to  $\mathcal{P}^\infty(\Lambda\text{-Mod})$  and is a  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -phantom of  $S$ . In fact, the map  $f : H \rightarrow S$  makes  $H$  an effective  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -phantom with respect to the class  $S^\infty(\Lambda\text{-mod})$  of all string modules of finite projective dimension.*

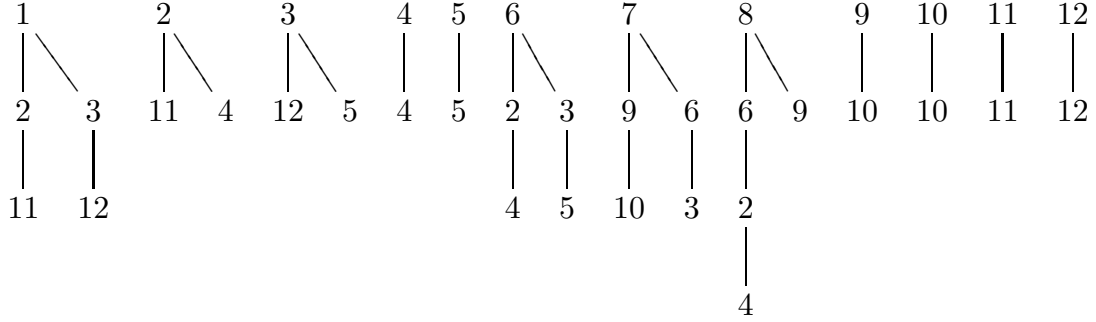
(III)  *$H$  is left and right periodic and can be constructed from  $\Gamma$  and  $I$  in fewer than  $3|\Gamma_0|$  steps.  $\square$*

Given a simple module  $S = \Lambda e/Je$  over a string algebra  $\Lambda$ , we will call the module  $H = H(S)$  of the theorem the *characteristic*  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -phantom of  $S$ . After illustrating the theory with a first example, we will give an inductive description of the finite segments of a graph

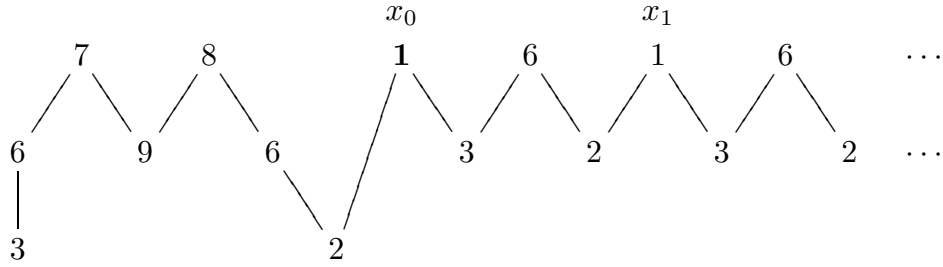


of the characteristic  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -phantom of  $S$ . Sending the top element marked  $x_0$  to  $e + Je$  in  $S$  and sending the other top elements to zero will then yield a map  $f : H \rightarrow S$  as stipulated in the theorem.

**Example F.** Let  $\Lambda = K\Gamma/I$  be the string algebra with the following indecomposable projective left  $\Lambda$ -modules



Then the characteristic  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -phantom  $H_1$  of  $S_1$  has graph



and the homomorphism  $f : H_1 \rightarrow S_1$  which sends  $x_0$  to a nonzero element of  $S_1$  and the  $x_i$  for  $i \geq 1$  to zero has the properties described in Theorem 9. In particular,  $S_1$  does not have a  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximation.

The characteristic  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -phantom of  $S_7$ , on the other hand, has graph



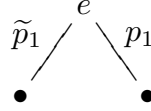
and thus coincides with the minimal  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximation of  $S_7$ . These graphs are obtained via the algorithm which we describe next.  $\square$

Recall that we refer to a module  $X$  as a top-embeddable submodule of  $Y$  if there exists a monomorphism  $f : X \rightarrow Y$  which induces a monomorphism  $X/JX \rightarrow Y/JY$ . Dually, we call  $X$  a *socle-faithful* factor module of  $Y$  if there exists an epimorphism  $f : Y \rightarrow X$  which induces an epimorphism  $\text{soc } Y \rightarrow \text{soc } X$ .

**Description of the characteristic phantom of a simple module  $S \in \Lambda\text{-mod}$ , where  $\Lambda$  is a finite dimensional string algebra.**

Let  $S = \Lambda e / J e$ . The following are the steps of an algorithmic procedure for constructing  $H = H(S)$ , but here we will not discuss the algorithmic nature, nor prove that the quantities stipulated in the process exist.

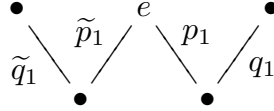
Step 1. Let  $p_1$  and  $\tilde{p}_1$  be paths starting in  $e$  which have *minimal* lengths  $\geq 0$  such that



is the graph of a string module which can be top-embedded into an object in  $\mathcal{S}^\infty(\Lambda\text{-mod})$ . In particular, we have  $p_1 \neq \tilde{p}_1$  unless both of these paths are trivial, and  $\text{startarrow}(p_1) \neq \text{startarrow}(\tilde{p}_1)$  if both are nontrivial.

If both  $p_1$  and  $\tilde{p}_1$  are trivial, we set  $H = S$ . Otherwise, we proceed to

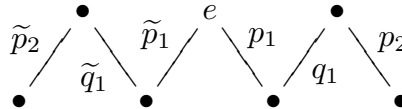
Step 2. Let  $q_1$  and  $\tilde{q}_1$  be paths ending in  $\text{end}(p_1)$  and  $\text{end}(\tilde{p}_1)$ , respectively, and having *maximal* lengths  $\geq 0$  with the property that



is the graph of a string module which arises as a socle-faithful factor module of an object in  $\mathcal{S}^\infty(\Lambda\text{-mod})$ . In case  $p_1$  is trivial, set  $q_1 = e$ , and deal symmetrically with  $\tilde{q}_1$ .

If both  $q_1$  and  $\tilde{q}_1$  are trivial, i.e., if the graph of Step 2 coincides with that of Step 1, we let  $H$  be the string module having this graph. Otherwise, we proceed to

Step 3. Let  $p_2$  and  $\tilde{p}_2$  be paths starting in  $\text{start}(q_1)$  and  $\text{start}(\tilde{q}_1)$ , respectively, which have *minimal* lengths  $\geq 0$  with the property that



is the graph of a string module which can be top-embedded into some object in  $\mathcal{S}^\infty(\Lambda\text{-mod})$ , ... etc.

After fewer than  $3|\Gamma_0|$  steps, this procedure either terminates or has become periodic on both sides. It is easy to recognize when one has hit a left or right period: Indeed, if  $\text{startarrow}(p_i) = \text{startarrow}(p_j)$  for some  $i < j$ , then  $p_{i+r} = p_{j+r}$  and  $q_{i+r} = q_{j+r}$  for all  $r \geq 0$ , the symmetric criterion holding for the other side.  $\square$

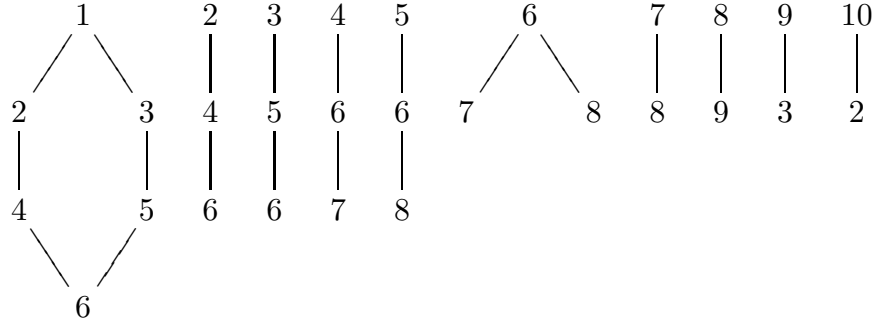
As one gleans from this inductive description of the phantoms  $H = H(S)$  of the simple modules  $S$ , string algebras merit their name also from a homological viewpoint. Indeed, the phantoms  $H(S)$  depend only on the string modules of finite projective dimension, and the  $K$ -dimensions of the  $H(S)$  in turn determine whether or not  $\mathcal{P}^\infty(\Lambda\text{-mod})$  is contravariantly finite. One may wonder whether our emphasis on string modules is just dictated by convenience and whether band modules play a similar homological role. They do not. In fact, there exist string algebras of positive little finitistic dimension which have no nontrivial (finitely generated) band modules of finite projective dimension. Another

asset of string algebras that arises as a byproduct of our main theorem we record somewhat more formally.

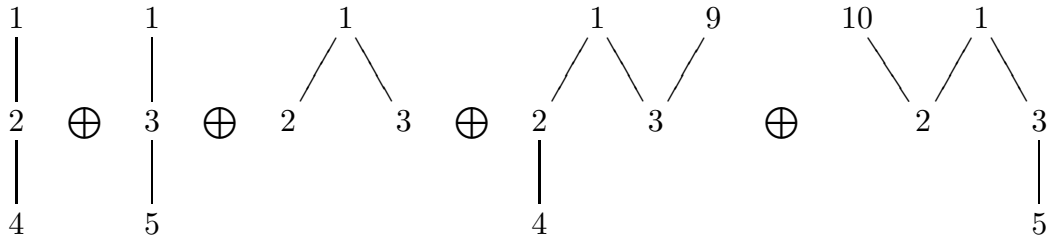
**Corollary 10.** *Suppose that  $S$  is a simple module over a finite dimensional string algebra  $\Lambda$ . If  $S$  has a  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximation, then the minimal  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximation of  $S$  is a string module and, in particular, is indecomposable.  $\square$*

The corollary does not carry over to arbitrary special biserial algebras, as the next example demonstrates.

**Example G.** Let  $\Lambda = K\Gamma/I$  be the special biserial algebra with indecomposable projectives



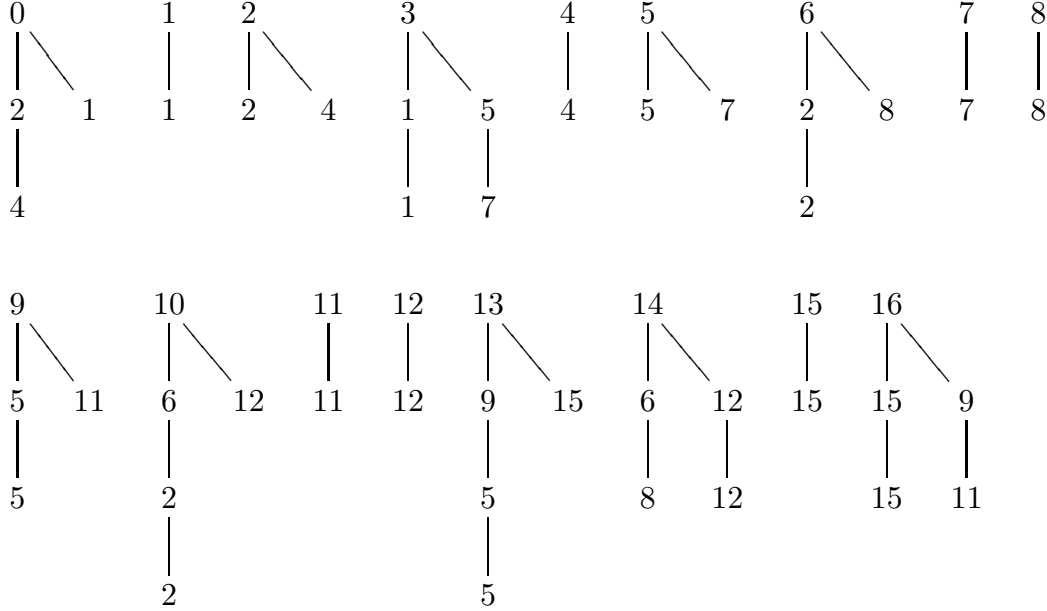
Then the minimal  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -approximation of  $S_1$  is as follows:



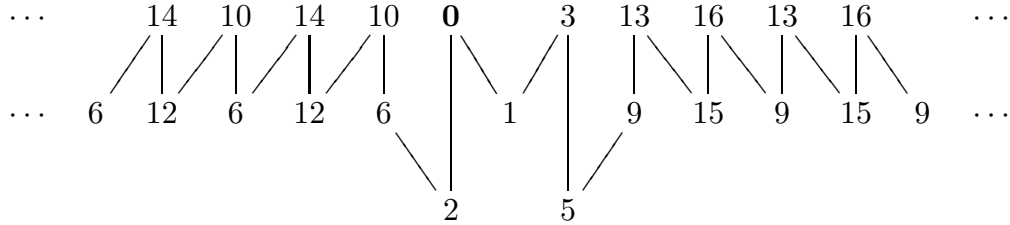
In particular, we observe that the factor modules of  $\Lambda e_1$  contained in the two rightmost summands are not minimal with respect to top-embeddability into objects of  $\mathcal{P}^\infty(\Lambda\text{-mod})$ . Thus the alternation “choose a minimal factor module embeddable into a module in  $\mathcal{P}^\infty(\Lambda\text{-mod})$ , then choose a maximal essential extension arising as a factor module of a module in  $\mathcal{P}^\infty(\Lambda\text{-mod})$ ” which leads to the minimal approximations of the simples over string algebras in case of existence, cannot be expected to achieve this goal in the more general situation.  $\square$

To give another illustration of our algorithm, we conclude with an example of a string algebra  $\Lambda$  and a simple left  $\Lambda$ -module  $S$ , the characteristic phantom of which is twosided infinite with left/right periods reached at different steps of the algorithm.

**Example H.** Let  $\Lambda = K\Gamma/I$  be the string algebra with the following indecomposable projective left modules:



Then the characteristic  $\mathcal{P}^\infty(\Lambda\text{-mod})$ -phantom of  $S = \Lambda e_0/Je_0$  has a graph as follows:



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